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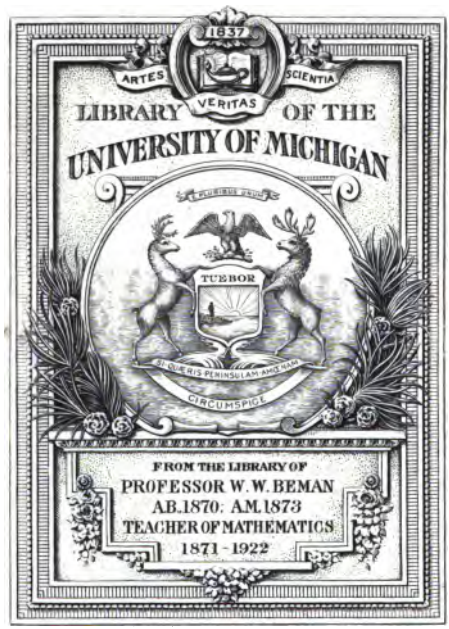
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4035. (Professor Wolstenholme, M.A., Sc.D.)—If, in any curve, PQ be a chord normal at P, then, when PQ is of maximum or minimum length, either PQ is normal also at Q, or Q is the centre of curvature at P 153

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$$\frac{x}{1!}+\frac{x^4}{4!}+\frac{x^7}{7!}+\ldots+\frac{x^{3n-2}}{(3n-2)!}+\ldots \text{ ad inf.}$$

is

$$\tfrac{1}{3}\left\{e^x+\frac{2}{e^{\frac{1}{2}x}}\sin\tfrac{1}{2}(x\sqrt{3}-\tfrac{1}{3}\pi)\right\},$$

and show what the series becomes when $x = \frac{1}{3}\pi\sqrt{3}$. 153

4333. (R. Tucker, M.A.)—Solve the equations

$$x^2+y^2+z^2-xy+yz-xz = a = 15,$$
$$x^2+y^2+z^2-xy-yz+xz = b = 9,$$
$$x^2+y^2+z^2+xy-yz-xz = c = 7. \quad \ldots\ldots 155$$

4434. (The Editor.)—Given the base and vertical angle, construct a triangle such that the rectangle under the sides shall have a given ratio to the square of a line drawn from the vertex to a fixed point in the base 155

4552. (Professor Crofton, F.R.S.)—If x be a first approximation to the n^{th} root of a number N, prove (1) that the root will be

$$\frac{(n+1)\,N+(n-1)\,x^n}{(n-1)\,N+(n+1)\,x^n}\,x, \text{ nearly,}$$

and find (2) whether this is a better approximation than Newton's, viz.,

$$\frac{N+(n-1)\,x^n}{nx^{n-1}}. \quad \ldots\ldots 156$$

4620. (The Editor.)—Two straight lines at right angles to one another being given in position, it is required to find in one of them a

point, such that its distance from the other, measured along a line drawn through itself and a given point, shall have a given ratio to its distance from a given point in the other line. 157

5216. (For Enunciation, see Question 3713.)..... 150

5430. (The Editor.)—Find the average length of the radius of curvature of a parabola, for points situated between the vertex and the latus-rectum. 158

5431. (The Editor.)—Supposing the Hugli river to flow due South, prove that the pressure on the Western bank at the depth d would be increased by the change of latitude of the running water in the ratio $(gd + bv\omega \sin l) : gd$, where b is the breadth of the river, v its velocity, l the latitude, and ω the angular velocity of the earth about its axis.... 158

5434. (The Editor.)—Find the volume between $x = 0$ and $x = 2l$ of the solid bounded by the surface whose equation is

$$a(y^4 + z^4) - z^2(x^2 - 2ay^2 + 2e^2) - y^2(bx^2 + c^2x + e^3) = 0 \text{..... } 159$$

5798. (Professor Cochez.)—Construire un triangle rectangle, connaissant les sommes de chacun des côtés avec le perpendiculaire abaissée du sommet de l'angle droit sur l'hypotenuse. 159

5873. (The Editor.)—Prove that (1) the locus of the poles, with regard to the auxiliary circle, of tangents to the corresponding ellipse, is a similar ellipse with its minor axis equal to the major axis of the first ellipse; and (2) the locus of points taken in the same manner with regard to the latter ellipse is another ellipse similar to the first. 160

5949. (J. Young, B.A.)—If the equation of a conic referred to any tangent and normal as axes, which is of the form

$$ax^2 + 2hxy + by^2 + 2fy = 0,$$

is transformed to the tangent and normal at any other point on the curve, prove that, in addition to the usual invariants $a + b$, $ab - h^2$, there is in this case a third invariant, viz., af^2. 161

5964. (J. J. Walker, M.A., F.R.S.)—If P be any point on a plane curve, O the pole, ON the line from which θ is measured, and PN the normal to the curve at P, prove that ON $= (r\,dr) / d(r\cos\theta)$. 163

6069. (The Editor.)—Find the locus of the centres of circles tangential to a parabola and the tangent at its vertex. 162

6084. (For Enunciation, see Question 5964.)..... 163

6127. (Professor Brahy.)—Si l'on coupe un hyperboloïde à une nappe par un plan et que, par tous les points de la courbe d'intersection, on mène des plans tangents à l'hyperboloïde, quelle sera la surface enveloppée des plans tangents. 164

6263. (J. Young, B.A.)—Show that the solution of Quest. 5798, given on p. 56 of Vol. XXXI., is incorrect. 159

6706. (For Enunciation, see Question 3713.)..... 150

6903. (W. J. C. Sharp, M.A.)—Find the law of density, varying as the depth, that the centre of pressure of a semi-circle, just immersed with its diameter horizontal, may divide the vertical radius as $m : n$. 164

6991. (J. J. Walker, M.A., F.R.S.)—Let A, B be points on a line determined by $(a, b, c\overline{\Sigma}x, 1)^2 = 0$; C, D two other points on the

same line determined by $(a_1, b_1, c_1 \mathbin{\textstyle\bigcirc} x, 1)^2 = 0$; prove that the values of the double, or cross, ratios AC . DB : AB . DC, AD . CB : AB . CD are the roots of $4\Pi' (2\lambda - 1)^2 - J^2 = 0$, where I, I', J are the invariants of the quadratic. 165

7156. (A. McMurchy, B.A.) — If $x = e^{\theta} \cos\theta$, $y = e^{\theta} \sin\theta$, prove that $\dfrac{d^m}{dx^m} \cdot \dfrac{d^n}{dy^n} \cdot \rho = -(m+n-1)!\, e^{-(m+n)\rho} \cos[(m+n)\theta + \frac{1}{2}\pi(2m+n)]$,

or $(m+n-1)\, e^{-(m+n)\rho} \sin[(m+n)\theta + \frac{1}{2}\pi(2m+n-1)]$. 47

7212. (D. Edwardes.)—Prove that

$$(y+z-2u)(z+x-2v)(x+y-2w) - (y+z-2u)(x+y-v-w)^2 - (z+x-2v)(y+v-w-u)^2 - (x+y-2w)(z+w-u-v)^2 = 2(x+u-v-w)(y+v-w-u)(z+w-u-v). \text{ ..... } 166$$

7528. (W. J. C. Sharp, M.A.)—Deduce HERMITE'S law of reciprocity from the identity

$$\frac{(1-x^{n+1})(1-x^{n+2}) \dots (1-x^{n+\theta})}{(1-x)(1-x^2) \dots (1-x^{\theta})} \equiv \frac{(1-x^{\theta+1})(1-x^{\theta+2}) \dots (1-x^{n+\theta})}{(1-x)(1-x^2) \dots (1-x^n)}.$$

..... 43

7538. (Professor Haughton, F.R.S.)—Show that the law of propagation of heat in a solid sphere is

$$\frac{dv}{dt} = a\left(\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr}\right). \text{ ..... } 167$$

7599. (Professor Wolstenholme, M.A., Sc.D.)—If

$$(1 + 2x\cos\alpha + x^2) / (1 - 2x\cos\alpha + x^2) \equiv z,$$

prove that

(1) $\displaystyle\int_0^1 (\log x)^6 \log z \frac{dx}{x} = \tfrac{1}{2^8}\pi\,(\tfrac{1}{2}\pi - \alpha)(17\pi^6 + 34\pi^4 u + 18\pi^2 u^2 + 4u^3)$,

(2) $\displaystyle\int_0^1 (\log x)^8 \log z \frac{dx}{x} = \tfrac{1}{5}\pi\,(\tfrac{1}{2}\pi - \alpha)(31\pi^8 + 62\pi^6 u + 33\pi^4 u^2 + 8\pi^2 u^3 + u^4)$,

where $u = \alpha(\pi - \alpha)$, and α lies between 0 and π; (3) in general,

$$\int_0^1 (\log x)^{2n} \log z \frac{dx}{x} = 4 \cdot 2n! \left\{ \frac{\cos\alpha}{1^{2n+2}} + \frac{\cos 3\alpha}{3^{2n+2}} + \frac{\cos 5\alpha}{5^{2n+2}} + \dots \right\},$$

if α be any angle from 0 to π, and this can always be expressed in the form

$$\pi(\tfrac{1}{2}\pi - \alpha)\left(A_0\pi^{2n} + A_1\pi^{2n-2}u + \dots + \frac{u^{2n}}{2n+1}\right). \text{ ..... } 38$$

7624. (Nilkantha Sarkar, B.A.)—If a train, consisting of p carriages, each of which will hold q men, contains $pq - m$ men, find the chance that another man getting in, and being equally likely to take any vacant place, will travel in the same carriage with a given passenger. 30

7645. (J. W. L. Glaisher, F.R.S.) — Show that (1) Mr. RAWSON'S expression in Question 7416 (Vol. XLI., pp. 50 and 51), consists of a series which proceeds to infinity both ways, and is divergent for all values of x; (2) this expression, regarded as a solution of the differential equation, is therefore without meaning until such a value is given to m or n as will destroy the second series; and (3), when this is done by

putting m = an uneven integer or n = an uneven integer, we obtain as two particular integrals the series U and V, and therefore ω AU + BV is the general integral of the differential equation. 39

7704. (Asûtosh Mukhopâdhyây, M.A., F.R.A.S.) — Prove that the loci of the middle points of series of parallel chords of the Witch of Agnesi, $xy^2 = 4a^2(2a-x)$, are, *in general*, curves of the third order; (2) the principal axis is an exception; and (3) explain how the locus degenerates into a hyperbola, and gradually into the asymptotes..... 34

7731. (J. McDowell, M.A.)—On the three sides of a triangle ABC, segments of circles are described, internally, capable of containing angles respectively equal to the supplements of the opposite angles, and intersecting in P. Find the lengths of the straight lines PA, PB, PC in terms of the sides and angles of the given triangle. 46

7786. (D. Biddle.)—Three trains in motion, A, B, C, occupy three parallel sets of rails, and their several lengths are 150, 130, and 100 yards. A and B are up-trains travelling at uniform speeds of 20 and 25 miles an hour respectively. B has overtaken A, and 35 seconds have passed since they began to overlap, when the head of C, a down train, passes the head of A. C is slackening speed, and comes to rest 55 seconds afterwards, and 350 yards from the point indicated. Find (1) the period of time during which all three trains are side by side, and give the positions of commencement and finish of such juxtaposition, supposing C to slacken speed at the same rate that a body projected vertically from the earth would if *in vacuo*, the distances traversed being, when measured from the point of rest, as the squares of the times occupied in traversing them; and (2) determine also the whole extent of permanent way covered at one time or other, during the transit, by the triple rectangular juxtaposed mass, as well as the maximum length of such mass, and the limits between which its variable centre oscillates. 82

7829. (J. Brill, B.A.)—If ABC be a plane triangle, and O a point within it; prove that

(1) $OB \cdot OC \cdot BC \cos(OBA - OCA) + OC \cdot OA \cdot CA \cos(OCB - OAB) + OA \cdot OB \cdot AB \cos(OAC - OBC) = BC \cdot CA \cdot AB.$

(2) $OA^2 \cdot BC \cdot \cos(OAB - OAC) + OB^2 \cdot CA \cos(OBC - OBA) + OC^2 \cdot AB \cos(OCA - OCB) = BC \cdot CA \cdot AB.$ 94

7831. (Asûtosh Mukhopâdhyây, B.A.)—Show that

$$u \equiv \sin\theta + \tfrac{1}{3}\sin 5\theta + \tfrac{1}{5}\sin 9\theta + \dots \text{ ad inf.} = \tfrac{1}{4}\log\frac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta}. \quad \dots \quad 90$$

7834. (W. J. C. Sharp, M.A.)—If lines be drawn from the vertices of any tetrahedron ABCD through the same point P, to meet the opposite faces in A′, B′, C′, D′ respectively: show that

$$\frac{PA'}{AA'} + \frac{PB'}{BB'} + \frac{PC'}{CC'} + \frac{PD'}{DD'} = 1. \quad \dots \quad 41$$

7881. (Sarah Marks.) — If mn balls have been distributed into m bags, n into each, find (1) the chance that two specified balls will be found in the same bag, and (2) what the chance becomes when r bags have been examined and found not to contain either ball..... 36

7883. (J. O'Regan.)—A and B play for a stake which is to be won by him who makes the highest score in 4 throws of a die. After two throws, A has scored 12, and B 9; find A's chance of winning. 32

7894. (Professor Hudson, M.A.)—Prove that, in the steady motion in one plane of a uniform incompressible fluid under the action of natural forces, if u, v be the velocities at x, y, parallel to the axes,

$$v\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)u-u\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)v=0. \ldots\ldots\ldots\ldots 167$$

7896. (The Editor.)—In a circle of radius a, a chord QRQ′ moves perpendicularly to a chord AX, whose central distance (CS) is $a\sin\alpha$, and from A tangents AP, AP′ are drawn to circles on QR, Q′R as diameters; show that (1) the locus of the points P, P′ is a heart-shaped curve whose polar equation, with A as pole and AX as polar axis, is

$$r(1+4\tan^2\tfrac{1}{2}\theta)=2a(\cos\alpha+2\sin\alpha\tan\tfrac{1}{2}\theta);$$

(2) when AX is a diameter, this cardioid is $r(1+4\tan\frac{1}{2}\theta)=2a$, which cuts the circle at a point T (whose distance from A is $AT=\frac{2}{3}a$, and whose abscissa is $AD=\frac{8}{9}a$), has a maximum ordinate VE at a point V whose abscissa is $AE=(\cdot934)\,a$, extends to a distance $AZ=(\cdot078)\,a$ to the left of A, and contains an area equal to eight-ninths of the area of the circle; (3) the equations of the cardioid between (x, r) and (y, r) are

$$\frac{x}{r}=\frac{5r-2a}{3r+2a},\quad \frac{y}{r}=\frac{4r^{\frac{1}{2}}(2a-r)^{\frac{1}{2}}}{3r+2a};$$

(4) if PG be a perpendicular to QR, then $(RG : GQ)=4\,(AR : RX)$; (5) the locus of the intersection K of the tangents from A, X to the variable circle is an ellipse of which A, X are the foci and the semi-axes $\frac{4}{3}a$, $\frac{4}{9}a$; (6) the distance from K to either of the points of contact of these tangents is two-thirds of the radius of the given circle. 75

7906. (J. Brill, B.A.)—The tangents to a parabola at the points P, Q, R meet the tangent at the vertex in the points X, Y, Z, respectively; prove that

$$SP\,.\,YZ\,.\,\sin ASP+SQ\,.\,ZX\,.\,\sin ASQ+SR\,.\,XY\,.\,\sin ASR=0\ldots(1),$$

$$AS\,(SP\,.\,YZ\,.\,\cos ASP+SQ\,.\,ZX\,.\,\cos ASQ+SR\,.\,XY\,.\,\cos ASR)$$
$$=YZ\,.\,ZX\,.\,XY \ldots\ldots\ldots\ldots (2),$$

subject to the usual convention concerning the signs of lengths measured along the same straight line. 47

7908. (R. Knowles, B.A.) — Tangents are drawn from a point T (h, k) to meet the ellipse $a^2y^2+b^2x^2=a^2b^2$, centre C, in P and Q; prove that (1) $CP^2\sim CQ^2=[4a^2b^2\,(a^2-b^2)\,hk\,(a^2k^2+b^2h^2-a^2b^2)]^{\frac{1}{2}}/(a^2k^2+b^2h^2)^2$; (2) if T be on either axis, CP and CQ are equal. 59

7912. (D. Edwardes.)—Prove that

$$\left(x^2\frac{d}{dx}\right)^n\frac{1}{x^{n-1}}\left(x\frac{d}{dx}-n\right)^r y=x^{n+1}\left(x\frac{d}{dx}\right)^r\frac{d^ny}{dx^n}. \ldots\ldots 40$$

7916. (Asûtosh Mukhopâdhyây, M.A.)—If a comet move in an hyperbola whose semi-axis is a, eccentricity ae, show that its place at the

end of t seconds, after leaving the perihelion, is determined from the equations

$$t = \frac{P}{2\pi}(a^3)^{\frac{1}{2}}[e\tan\theta - \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta)], \quad \tan\tfrac{1}{2}V = \left(\frac{e+1}{e-1}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\theta,$$

where V = the true anomaly reckoned from the perihelion, P = the earth's periodic time, her mean distance being unity..... 36

7978. (Sarah Marks.)—If O be the in-centre, O′ the circum-centre and P the ortho-centre, R the circum-radius of the triangle ABC, prove that the area (Δ_1) of the triangle OO′P is

$$\Delta_1 = -2R^2 \sin\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(C-A)\sin\tfrac{1}{2}(A-B)..... 44$$

7979. (R. Knowles, B.A.)—If Δ be the area of the triangle ABC, and Δ_1 that formed by joining the ex-centres, show that

$$\Delta_1 : \Delta = 8sR^2 : abc. ..... 42$$

8000. (Professor Wolstenholme, M.A., Sc.D.)—In a cubic with three real asymptotes and an acnode, prove that the area between any branch of the curve, and the two asymptotes which include it, is one-third of the area of the triangle formed by the asymptotes. [The equations of the curve in Cartesian and Areal coordinates are

$$27xy\left(1 - \frac{x}{a} - \frac{y}{b}\right) = ab, \; 27xyz = (x+y+z)^3.] ..... 26$$

8032. (Satis Chandra Ray.)—If S be the area of any triangle, and S_n that of its n^{th} pedal, prove that

$$\frac{S_n}{S} = (-2)^{2n}\sin(2^n-1)A \,.\, \sin(2^n-1)B \,.\, \sin(2^n-1)C..... 45$$

8034. (Asûtosh Mukhopâdhyây, M.A., F.R.A.S.)—If

$$x = [y(y-\lambda)]^{\frac{1}{2}} \pm \lambda \log \frac{[\sqrt{y-\lambda}]^{\frac{1}{2}} + [\sqrt{y}+\sqrt{\lambda}]^{\frac{1}{2}}}{[\sqrt{y-\lambda}]^{\frac{1}{2}} - [\sqrt{y}+\sqrt{\lambda}]^{\frac{1}{2}}},$$

prove that $\left(\frac{d^2r}{dy^2}\right)^2 = \frac{\lambda^2}{4}\cdot\frac{1}{y(y-\lambda)^3}$ or $\frac{\lambda^2}{4}\cdot\frac{1}{y^3(y-\lambda)}$. 45

8035. (Rev. T. C. Simmons, M.A.)—If three lengths be chosen at random, prove that they are just as likely as not to denote the sides of a possible triangle. 88

8048. (Professor Cochez.)—Trouver la somme

$$\frac{1}{\cos x\cos y} + \frac{1}{\cos y\cos z} + \ldots \frac{1}{\cos t\cos u} + \frac{1}{\cos u\cos w},$$

$x, y, z \ldots t, u, w$ étant en progression arithmétique, ρ étant la raison. 92

8072. (Asûtosh Mukhopâdhyây, M.A., F.R.A.S.)—Prove that,

$$\int_b^a \phi(x)\,dx = \frac{a-b}{\alpha-\beta}\int_\beta^\alpha \phi\left(\frac{a-b}{\alpha-\beta}x + \frac{ab-\beta a}{\alpha-\beta}\right)dx. ..... 41$$

8077. (Asûtosh Mukhopâdhyây, M.A., F.R.A.S.)—Upon two lines AB, AB′, two points C, C′ are taken, such that AC . AC′ = BC . B′C′. Find the curve which is constantly touched by a straight line passing through C, C′. 42

8084. (Professor Sáradáranjan Ráy, M.A.)—If P_n be a zonal harmonic of the n^{th} order, prove that

$$P_n = C(1-\mu^2)^{\frac{1}{2}n} + n(1-\mu^2)^{\frac{1}{2}n}\int_0^\mu P_{n-1}(1-\mu^2)^{-\frac{1}{2}(n+2)}\,d\mu,$$

where $C = 0$ if n be odd, and $= (-1)^{\frac{1}{2}n}\, n!\, 2^{-n}[\frac{1}{2}n!]^{-2}$ if n be even.... 43

8112 & 8147. (R. Knowles and the Rev. J. J. Milne, M.A.)—Given two circles which do not intersect; from any point on the one, tangents are drawn to the other: prove that the envelope of the chord of contact and the locus of its middle point are respectively a hyperbola and its auxiliary circle. 61

8126. (Professor Hudson, M.A.)—In sailing in a great circle from a port in north latitude λ to another in north latitude $\lambda - \delta$, the difference of longitude being 2ϕ, show that part of the shorter course will lie further north than the northernmost port, unless δ exceed

$$\tan^{-1}\left[\sin 2\lambda \sin^2\phi \,/\, (1-2\sin^2\lambda\sin^2\phi)\right]. \quad \ldots\ldots\ldots 50$$

8146. (W. J. McClelland, B.A.)—If the pairs of opposite angles of any spherical quadrilateral be denoted by A and A', B and B', C and C', and the diagonals by δ_1, δ_2, δ_3; prove the *algebraic* relation

$$\sin A \sin A' \cos\delta_1 + \sin B \sin B' \cos\delta_2 + \sin C \sin C' \cos\delta_3 = 0,$$

and deduce from it as particular cases the following well-known theorems:—(1) The perpendiculars of a spherical triangle are concurrent; (2) if two diagonals of a spherical quadrilateral are quadrants, the third diagonal is also a quadrant. 38

8147. (For Enunciation, see Question 8112.)..... 61

8154. (W. J. C. Sharp, M.A.)—If planes be drawn touching the sphere, circumscribed about a given tetrahedron, at its vertices, these will contain another tetrahedron: determine its angles, plane, and dihedral..... 43

8156. (Professor Sylvester, F.R.S.)—If two non-intersecting edges of a pyramid and the cosine of their inclination be multiplied together, prove that one of the three products thus formed is equal to the sum of the other two. Show also that the shortest distance between any given pair of non-intersecting edges of a pyramid is equal to the quotient of its content by the square root of a quadratic function of its six squared edges. 29

8159. (Professor Wolstenholme, M.A., Sc.D.) — Two circles with centres A, B have radii a, b, and the distance between their centres c, and $a > b$; a straight line moves so that the segments cut from it by the two circles have a given ratio $\lambda : 1$; prove that (1) the envelope will be an ellipse whose major axis lies along AB if λ be < 1 or $> a/b$; and an hyperbola whose transverse axis is along AB if $\lambda > 1$ and $< a/b$; and that, if S, S' be the foci of the envelope, $AS' : AS = S'B : BS = \lambda + 1 : \lambda - 1$, so that S, S' divide AB harmonically, and if C be the centre of the envelope $CA = \lambda \,.\, CS$; (2) when $\lambda = 1$, the envelope is a parabola whose focus bisects AB, whose tangent at the vertex is the radical axis of the two circles, and latus rectum is $2(a^2 - b^2)/c$; (3) when $\lambda = a/b$, the envelope reduces to a point pair, the centres of similarity,—when λ is given by the equation $(a^2 - \lambda^2 b^2)(\lambda^2 - 1) = \lambda^2 c^2$, which necessitates $a > b + c$, there is only one straight line which satisfies the condition, and the envelope reduces to this straight line $x = \lambda^2 c/\lambda^2 - 1$; (4) if we assume λ a pure imaginary $\mu\sqrt{(-1)}$,

we get for the envelope an ellipse or hyperbola whose conjugate axis lies along the line of centres, the envelope can never become a parabola or a point pair, but may reduce to a single straight line when

$$\mu^2c^2 = (a^2+\mu^2b^2)(\mu^2+1), \quad (c > a+b);$$

and this equation gives the maximum and minimum values of μ, but they cannot be considered geometrical. 65

8161. (Émile Vigarié.)—On donne un cercle O de diamètre AB, deux tangentes parallèles AP, BQ, une troisième tangente mobile PQ et un point fixe C sur le diamètre. On abaisse OD, OE respectivement perpendiculaires sur PC, QC; la droite DE coupe AB en F. Enfin on élève CG perpendiculaire à AB; cette droite coupe la circonférence en G. Démontrer que (1) le point F est fixe; et (2) la droite GF est la médiane antiparallèle du triangle GOC. 53

8165. (The Editor.) — If from the ends (A, A′) of the major axis AA′ of an ellipse, tangents (AQT, A′RT) are drawn to a circle on an ordinate PM to this axis as diameter, prove that the distance of the point (T) of intersection of these tangents from either of their points (Q, R) of contact is constant, and has to the axis of the ellipse the ratio $1-e^2 : 3+e^2$, where e is the eccentricity of the ellipse. 29

8167. (The Editor.) — Prove that the Symmedian point of any triangle—viz., the cointersection of the lines joining the mid-point of each perpendicular with the mid-point of the corresponding side—may be regarded as the centre of inertia of three masses proportional to a^2, b^2, c^2, placed at the angular points of the triangle. 57

8168. (S. Roberts, M.A.)—Let A, B be two given points. A series of circles touch the straight line joining A, B at B. Another series of circles pass through A and B. The locus of the variable intersection of a circle of one series with a circle of the other, when these have AC as a common tangent, is the inverse of a parabola whose axis is AB with respect to B as pole. Show this; and by means of the result determine the regions within which a point P′ must lie so that a triangle with a base, given in position and magnitude, may have the point P′, (1) for a Brocard-point of either species; (2) of one species. 23

8169. (R. Tucker, M.A.) — P is any point in the triangle ABC; PA, PB, PC are cut in D, E, F, so that PD = k . PA, PE = k . PB, PF = k . PC; prove that, if O, O′ be the circum-centres of ABC, DEF, P, O, O′ are collinear, and P, O′ describe similar curves. 34

8172. (John Griffiths, M.A.)—If n be an integer, show that

$$\mathrm{dn}\frac{K}{2n}\,\mathrm{dn}\frac{3K}{2n}\,\mathrm{dn}\frac{5K}{2n} = \mathrm{dn}\frac{2n-1}{2n}K = (k')^{\frac{1}{4}n},$$

k' being the modulus complementary to k. 22

8182. (R. Knowles, B.A., L.C.P.) — CP, CD are conjugate radii of the ellipse $b^2x^2+a^2y^2 = a^2b^2$; show that PD touches at its mid-point the ellipse $2b^2x^2+2a^2y^2 = a^2b^2$. 66

8187. (The late Professor Clifford, F.R.S.) — The quantities $\xi+i\eta$, $x+iy$ being connected by the equation $x+iy = dn(\xi+i\eta)$, find (1) the relation between x and y which makes ξ or η constant; and (2) show that the curves ξ = const. cut the curves η = const. at right angles; and draw figures indicating the general form of the curves. 122

8188. (Professor Longchamps.)—Résoudre et discuter l'équation
$$\sin(45^\circ + \tfrac{3}{2}x) = h \sin(45^\circ - \tfrac{1}{2}x).$$ 49

8192. (Professor Nilkantha Sarkar, M.A.)—Given the perpendiculars from the angular points of a triangle on the opposite sides, construct the triangle. 63

8193. (Professor Hudson, M.A.)—A cone, with vertex up and closed with a plane horizontal base, contains $\frac{1}{16}$ of its volume of water, and rotates with uniform angular velocity about a vertical axis: show that, when the angular velocity is such that the free surface touches the base of the cone, it bisects the slant side. 62

8195. (The Editor.)—If n men pay equally towards the purchase of a grindstone of radius a, and respective thicknesses $2b$, $2c$ at circumference and centre, show (1) that if the rate of decrease of thickness is uniform, and $m = (b-c)/a$, the radius (x) of the portion of the stone remaining for the n^{th} man to grind down for his share will be given by the equation $nx^2(2mx+3c) = a^2(2b+c)$; (2) apply this to find what part of a stone, 24 inches in radius, 6 inches thick at circumference, and 4 inches at centre, must be ground down by each of 3 men paying equally; and solve the problem when the grindstone is of the form of a double concave lens, (3) spherical, or (4) paraboloidal. 51

8200. (Rev. T. C. Simmons, M.A.)—Two straight lines are drawn at random in a given plane. Is the chance of their parallelism infinitesimal or zero? 53

8213. (R. Knowles, B.A.)—If tangents TP, TQ from $T(x, y)$ touch the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ in P, Q, show that
$$TP \cdot TQ = \frac{(a^2y^2 + b^2x^2 - a^2b^2)\left[(x^2+y^2)^2 + (a^2-b^2)(a^2-b^2-2x^2+2y^2)\right]^{\frac{1}{2}}}{a^2y^2 + b^2x^2}.$$
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8216. (Professor Sylvester, F.R.S.)—Let x, y, z, t be the distances of any point P, and x', y', z', t' of P′, from the faces of a fundamental pyramid divided respectively by the distances of the vertices a, b, c, d of the pyramid from those faces; prove that $PP'^2 = \Sigma(ab)^2(x-x')(y'-y)$. [Of course a precisely similar formula applies to distances on a plane or in space of any number of dimensions.] 21

8219. (Professor Wolstenholme, M.A., Sc.D.)—If $y^5 - 5y^3 + 5y = 2x$, prove (1) that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \frac{1}{25}y = 0$; and (2), by means of this equation, obtain the five expansions of y in series of positive ascending powers of x. 56

8224. (S. H. Boult, M.A.)—Express the Compound Interest on a sum of money for any number of years at any rate per cent. by a series... 21

8228. (Capt. P. A. MacMahon, R.A.)—If A, B, a, b, c be positive integers, and $100a + 10b + c$ be divisible by $10A + B$, prove that $cA^2 - bAB + aB^2$ is also so divisible. 33

8230. (Piers C. Ward, M.A.)—Show that the rationalization of $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} + d^{\frac{1}{3}} = 0$ is equivalent to the eliminant of x from
$$(x-a-l)^3 - 27abx = 0, \quad \text{and} \quad (x+c+d)^3 - 27cdx = 0. \ldots\ldots 58$$

8233. (R. Tucker, M.A.)—P is a point on an ellipse; PM, PN, perpendiculars on the equiconjugate diameters, are produced to meet their productions in M′, N′; the normal at P (which bisects MN) cuts MN in K and M′N′ in K′: find the loci of K, K′. 67

8234. (Rev. J. J. Milne, M.A.) — If the normal PG of a conic be given in magnitude and position, and also the eccentricity, prove geometrically that the *latus rectum* will envelope a conic of the same species. 31

8241. (D. Edwardes.)—Show that

$$\left(\frac{1}{1+n^2}+\frac{1}{3^2+n^2}+\frac{1}{5^2+n^2}+\&c.\right)\left(\frac{1}{n^2}+\frac{2}{2^2+n^2}+\frac{2}{4^2+n^2}+\&c.\right)=\frac{\pi^2}{8n^2}.$$

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8245. (Professor Neuberg.)—Trouver dans le plan d'un triangle ABC un point M tel que les perpendiculaires menées par M sur les bissectrices des angles A, B, C et limitées par ces angles soient égales entre elles. Trouver la longueur commune de ces droites. 22

8247. (Professor Hudson, M.A.) — If $S_r = 1^r+2^r+3^r+\ldots+n^r$, prove that $\frac{(n+1)^{r+1}-n^{r+1}-(n+1)+(-)^r n}{(r+1)\,r} = S_{r-1}+\frac{(r-1)(r-2)}{3.4}S_{r-3}$ $+\frac{(r-1)(r-2)(r-3)(r-4)}{3.4.5.6}S_{r-5}+\ldots$, ending with the term involving S_2 or S_1 according as r is odd or even. 64

8248. (Professor Nash, M.A.) — A candle, in a vertical position, is connected with a body of weight equal to its own, by a string passing over a smooth pulley, from which the weight and the candle hang freely. The candle is then lighted; prove that, if n seconds be the time for which it will burn, the weight will, in t seconds, have descended through a space

$$2ngt-\frac{gt^2}{2}+2ng\,(2n-t)\log\frac{2n-t}{2n}.$$ 37

8250. (Professor Macfarlane, F.R.S.E.)—Prove that

$$\frac{1}{2}+\frac{1}{2.4}+\frac{1.3}{2.4.6}+\ldots \text{ad inf.} = 1 \quad \ldots\ldots(1);$$

$$\frac{1}{2}-\frac{1.3}{2.4.6}+\frac{1.3.5.7}{2.4.6.8.10}-\ldots \text{ad inf.} = \left\{\frac{(2^{\frac{1}{2}}-1)}{2}\right\}^{\frac{1}{2}} \quad \ldots\ldots(2).$$

..... 28

8251. (Professor Matz, M.A.) — A candidate is examined in three papers, to each of which m marks are assigned as a maximum; show that, if his total on the three papers is $2m$, there are $\frac{1}{2}(m+1)(m+2)$ ways in which this may occur. 64

8255. (Rev. T. C. Simmons, M.A.)—A triangle is formed by three random tangents drawn to a given circle. Prove that the odds are 15 to 1 against its being acute-angled. 48

8261. (R. Knowles, B.A.)—From a point T tangents are drawn to a parabola whose focus is F; the tangent at the vertex meets these in p, q: prove that p, q lie on the circle described on TF as diameter. 34

8264. (S. Tebay, B.A.) — The shaft of a mine, 132 yards deep, is worked by two concentric grooved pulleys, each 9 feet diameter, revolving in opposite directions, and carrying a coiled rope $1\frac{1}{4}$ inches in diameter; find in what part of the shaft the cages pass each other. 32

8265. (W. J. Greenstreet, B.A.) — If P, Q, R are points on an ellipse, P′, Q′, R′ the ends of corresponding conjugate diameters, and circles, radii r, r', are drawn circumscribing triangles PQR, P′Q′R′; show that $r^2+r'^2=\rho^2+\rho'^2+a^2+b^2$, where ρ, ρ' are the distances of the centres of these circles from the centre of the ellipse. 60

8266. (Captain MacMahon, R.A. Generalised in Question 8400:)— Prove that the straight lines joining the mid-point of the side BC of the triangle ABC with the in-centre and the ex-centre to BC, meet the perpendicular from A on BC, each in a point distant from A a length equal to the radius of the corresponding circle. 91

8267. (Asûtosh Mukhopâdhyây, M.A., F.R.A.S.) — Evaluate in a finite form, $\int_D^{-na} \frac{r\,dr}{\left[2a(n-1)+\left\{r^2-4a^2(n-1)\right\}^{\frac{1}{2}}\right]^3}$. 107

8271. (Charlotte A. Scott, D.Sc.)—ABC is a triangle; lines $A\alpha\beta$, $B\beta\gamma$, $C\gamma\alpha$ are drawn, making equal angles respectively with AB, BC, CA. Show that (1) all the triangles so formed have one Brocard-point fixed, while the other describes a circle concentric with ABC; and (2) any point connected with the triangle describes a circle, while any straight line passes through a point..... 116

8273. (J. Brill, M.A.) — Prove that the locus of the centroid of the points in which a variable tangent to a parabola is intersected by n fixed tangents is also a tangent of the parabola, and its point of contact lies on the line joining the centroid of the $\frac{1}{2}n(n-1)$ points of intersection of the fixed tangents to the centroid of their points of contact, and divides that line in the ratio of $1:n-1$. 62

8275. (Professor Sylvester, F.R.S., & T. Muir, LL.D.)—Prove that

$$\begin{vmatrix} \lambda+aa+a'a' & ab+a'b' & ac+a'c', & \dots \\ \beta a+\beta' a' & \lambda+\beta b+\beta' b' & \beta c+\beta' c', & \dots \\ \gamma a+\gamma' a' & \gamma b+\gamma' b' & \lambda+\gamma c+\gamma' c', & \dots \end{vmatrix} = \lambda^{n-2}\left\{\lambda^2+\lambda(A+B)+AB-CD\right\};$$

where n = order of determinant,

$A=(a, b, c, \dots \sum a, \beta, \gamma, \dots)$, $B=(\alpha, \beta, \gamma, \dots \sum a', b', c')$,

$C=(a, b, c, \dots \sum a', \beta', \gamma', \dots)$, $D=(a', b', c', \dots \sum \alpha, \beta, \gamma)$.

This result, when $A=s=B$ and C or D = 0, becomes $\lambda^{n-2}(\lambda+s)^2$... 85

8278. (Professor Hudson, M.A.)—Prove that the locus of the point of intersection of normals to $y^2=4ax$ at the points where it is intersected by a tangent to the locus of the intersection of normals at right angles is $3375ay^2=16(x-5a)(x-50a)^2$. 54

8281. (Professor Byomakesa Chakravarti, M.A.) — A traveller in a railway carriage moving on a curve in a flat country is seated with his face towards the engine, and looks out on the inside of the curve. Show that the country beyond a certain point disappears, and that the

country nearer than this point comes into view; and that this point moves with a velocity $\frac{d\rho}{dt}\sin\alpha + v\cos\alpha$, where ρ is the radius of curvature of the curve, v the velocity of the train, and α the angle which the plane from his eye to the vertical edge, nearest the engine, of the window which bounds his view makes with the direction of the train's motion. 95

8282. (The Editor.) — If a coin be dropped at random over a grating composed of parallel equidistant wires in a horizontal plane, show that, according as the distance between the wires is equal to (1) the radius, (2) the diameter, (3) the circumference of the coin, the respective probabilities that the coin will not go through without striking, are

$$p_1 = \tfrac{1}{6}\pi + \tfrac{1}{4}\sqrt{3} = \cdot 9566,\ p_2 = \tfrac{1}{4}\pi = \cdot 7854,\ p_3 = \tfrac{1}{4} = \cdot 25,$$

so that the odds are, in (1) 22 to 1, and in (2) 11 to 3, in favour of striking, and in (3) 3 to 1 against striking. 113

8283. (Capt. H. Brocard.) — La parabole tangente aux deux bissectrices d'un angle A du triangle et aux deux hauteurs correspondant aux côtés de cet angle a pour équation

$$(\beta\cos C + \gamma\cos B)^2 + 4\alpha\cos A(\alpha\cos A - \beta\cos B - \gamma\cos C) = 0 \ldots 89$$

8287. (R. Tucker, M.A.)—DEF is the pedal triangle of ABC, whose Brocard-points are ω, ω', and Lemoine-point σ: the Brocard-points and Lemoine-points of AFE, BDF, CED are respectively ω_1, ω_1', σ_1, and so on; prove that (1) ω_1, ω_1'; ω_2, ω_2'; ω_3, ω_3' are concyclic with ω, ω' (*i.e.*, the three circles are coaxal); (2) if ρ', ρ_1', ρ_2', ρ_3' are the Brocard-radii of ABC and of AFE, &c., then $\rho_1'^2 + \rho_2'^2 + \rho_3'^2 = \rho'^2(1 - 2\cos A\cos B\cos C)$; (3) the following construction will give the Lemoine-point of AFE, &c.: — Produce the perpendicular $\kappa\sigma$ (on BC) to a point σ_1, such that $\kappa\sigma_1 = 2\kappa\sigma$; this is the point sought; (4) σ is the mean centre (or centroid) of σ_1, σ_2, σ_3. 57

8290. (J. Brill, M.A.) — Prove that in any triangle (n positive integer)

$$c^n = a^n\cos nB + na^{n-1}b\cos[(n-1)B - A] + \frac{n(n-1)}{2!}a^{n-2}b^2\cos[(n-2)B - 2A] + \ldots + b^n\cos nA \ldots\ldots (1);$$

$$\frac{c^n}{b^n}\cos nA = 1 + n\cdot\frac{a}{c}\cos B + \frac{n(n+1)}{2!}\cdot\frac{a^2}{c^2}\cos 2B + \ldots \text{where } a < c \ldots (2).$$

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8291. (W. J. McClelland, B.A.) — If a triangle ABC be inscribed in a circle of a co-axal system, the three sides each touching circles of the system at points X, Y, Z; prove that the lines AX, BY, CZ are concurrent. 97

8299. (Asûtosh Mukhopâdhyây, B.A., F.R.A.S.) — Integrate the equation $\left(\frac{d\theta}{d\phi}\right)^2 = \frac{\sin\theta}{\sin 2\phi}$. 81

8306. (Professor Sylvester, F.R.S.)—If the product of

$\rho(\cos\lambda + i\sin\lambda) - r(\cos\alpha + i\sin\alpha)$ by $\frac{1}{\rho}(\cos\lambda + i\sin\lambda) - r(\cos\alpha + i\sin\alpha)$,

is $P(\cos\omega + i\sin\omega)$, find the value of ω by a geometrical construction. 94

8307. (Professor Catalan.)—Démontrer que
$\cos^n x + \cos^{n-1} x \cos x + \cos^{n-2} x \cos 2x + \ldots + \cos nx = \sin(n+1)x \operatorname{cosec} x$.
..... 24

8309. (Professor Neuberg.) — Soient $2a'$, $2b'$ deux diamètres d'une ellipse, perpendiculaires à deux diamètres conjugués. Démontrer que le rapport $\frac{a^2b^2 - a'^2b'^2}{a^2+b^2-(a'^2+b'^2)}$ est constant. 107

8316. (Professor Clarke, F.R.S.) — Inscribe the maximum rectangle in a given parallelogram. 98

8317. (Professor Sáradáranjan Rây, M.A.)—The triangle formed by a chord of a given ellipse and the radii vectores drawn from the centre to its extremities is given in area; show that the locus of a point that divides the chord in a fixed ratio is a concentric and similar ellipse. 93

8318. (The Editor.)—If four points P, Q, R, S be taken at random, one in each side of a quadrilateral ABCD, find (1) the probability that the area of PQRS will be less than n (area of ABCD), where n is a proper fraction; (2) apply the result to the case of a rectangle; and therefrom (3) show that the probability that PQRS will be less than three-fourths (or, what is the same thing, greater than one-fourth) of ABCD is
$\frac{1}{2} + \frac{5}{4}\log_e 2 = \cdot 991434$. 103

8321. (For Enunciation, see Question 8275.) 85

8323. (Morgan Jenkins, M.A. Suggested by Quest. 7756.) — Show how to construct a triangle of given form (sides proportional to a, b, c), having given three lines (h, k, l) severally equal to the distances from its angular points to a point in the plane of the triangle. 67

8327. (W. J. C. Sharp, M.A.)—If a conic touch the three lines drawn from the intersections of a given straight line with the sides of a triangle to the opposite angles, prove that the intersections of the other three tangents from the vertices of the triangle with the opposite sides are collinear. 54

8348. (Professor Catalan.)—Un triangle ABC étant inscrit à un cercle, soient EDF le diamètre perpendiculaire au coté AB, et G la projection de C sur EF; la demi-somme des cotés AC, BC est moyenne proportionnelle entre les segments DF, GE; leur demi-différence est moyenne proportionnelle entre les segments DE, FG. 60

8349. (Professor Neuberg.) — Lorsque les sommets B, C d'un triangle ABC sont fixes, et que le sommet A décrit une droite donnée BX; (1) le centre de gravité et le centre du cercle des neuf points décrivent des droites, et la droite qui joint ces points enveloppe une parabole; (2) la symédiane partant de A enveloppe une conique ayant un foyer en C, et le centre des symédianes décrit une conique; (3) l'un des points de Brocard décrit une circonférence, et l'autre une podaire de conique. 49

8351. (Professor Hudson, M.A.) — Prove that the two generators of the hyperboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ through the point (x, y, z) will meet the principal elliptic section at the ends of diameters at right angles if
$\frac{x^2+y^2}{a^2+b^2} = \frac{z^2}{c^2}$. 64

8352. (Professor Sáradáranjan Rây, M.A.)—One angle A of a triangle is given in magnitude and position; if the base a also be given, show that the bisectors of the exterior base angles touch their envelopes at distances $2a \cos B \cos \frac{1}{2}C \operatorname{cosec} A$ and $2a \cos C \cos \frac{1}{2}B \operatorname{cosec} A$ from the respective base angles. 97

8353. (Professor Booth, M.A.)—Eliminate (1) x, y, z from the equations $ax^2 + 2hxy + by^2 = 0$, $by^2 + 2fyz + cz^2 = 0$, $cz^2 + 2gzx + ax^2 = 0$, and give (2) a geometrical interpretation of the result. 59

8362. (Maurice d'Ocagne.)—Si sur chacune des diagonales d'un trapèze prise pour diamètre on décrit un cercle, l'axe radical de ces cercles est perpendiculaire aux bases du trapèze et passe par le point de rencontre des côtés obliques. 93

8363. (H. G. Dawson, B.A.)—Solve the equations

$$x^2 - yz - zw - wy = a^2, \quad y^2 - zx - xw - wz = b^2 \ldots\ldots (1, 2),$$
$$z^2 - xy - yw - wx = c^2, \quad w^2 - xy - yz - zx = d^2 \ldots\ldots (3, 4).$$

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8366. (W. S. M'Cay, M.A.)—If two triangles have the same centre of gravity and have a centre of perspective at infinity, prove that their axis of perspective passes through their common centre of gravity... 91

8371. (Professor Purser, M.A.)—A variable circle touches two fixed circles, the character of the contacts being given. Show that its chord of intersection with any third fixed circle envelopes a conic section..... 85

8374. (W. Voyson, B.A.)—If through O_2, O_3, two of the ex-centres of a triangle ABC, lines $O_2b_2c_2$, $O_3b_3c_3$ be drawn parallel to BC, meeting AC, AB in b_2, b_3, c_2, c_3 respectively, prove that the line c_3b_2 will pass through the in-centre and c_2b_3 through the third ex-centre. 124

8375. (Rev. T. C. Simmons, M.A.) — If from ω, ω', the respectively *positive* and *negative* Brocard-points of a triangle ABC (whose Brocard-angle is θ), perpendiculars be drawn on the sides, thereby forming two pedal triangles DEF, D'E'F', then (1) these triangles are similar to ABC and equal to one another; prove also (2) that they have a common circum-circle, the triangular coordinates of whose centre, lying midway between ω and ω', are proportional to $a^2(b^2 + c^2)$, $b^2(c^2 + a^2)$, $c^2(a^2 + b^2)$; (3) that ω is the *positive* Brocard-point of DEF, and the centre of similitude of ABC, DEF; likewise ω' the *negative* Brocard-point of D'E'F'; (4) that the other Brocard-points of DEF, D'E'F' lie on the circle described on $\omega\omega'$ as diameter; (5) that the angles which the sides of DEF, D'E'F' make with corresponding sides of ABC are all equal to $\frac{1}{2}\pi - \theta$; (6) that the ratio of similitude of the triangles DEF, ABC, or of D'E'F', ABC is equal to $\sin\theta$; (7) that the triangles AFE', BDF', CED' are all similar to ABC, and the lines DE', EF', FD' respectively parallel to AB, BC, CA; (8) that the intercepts DD', EE', FF' on the sides are respectively proportional to $\sin C \sin(C - A) + \sin B \sin(B - A)$, and two similar expressions; (9) that if R be the circumradius, and ρ the "T. R." radius of ABC, and if R' be the circumradius of DEF or D'E'F', then $R'^2/R^2 + R^2/4\rho^2 = 1$.

[In the absence of a better means of distinguishing between the Brocard-points ω, ω' of a triangle ABC, Mr. Simmons suggests the following:—Joining ω to B, suppose the Brocard-angle given by ωBC. Then, if in moving from ω to B and therefrom to C we revolve round the angle ωBC in what is known as the positive direction, call ω the *positive*

Brocard-point; if in a negative direction, call ω the *negative* Brocard-point. The above circle, if thought worthy of a name, may be called the *sine-circle* of the triangle, since its radius $= R \sin\theta$.] 98

8381. (John Griffiths, M.A.)—If n be a number > 1 and s a number from 1 to $n-1$, show that

$$1 + 2\Sigma(-)^s \cos^2 s\pi / 2n = 0 \text{ ..... } 109$$

8383. (A. F. Theodosius.)—Prove that the radical centre of the three escribed circles of a triangle lies on the straight line joining the centre of the inscribed circle of the triangle to its centre of gravity. 122

8384. (W. J. McClelland, M.A.)—If t_1, t_2, t_3 be the lengths of the tangents from the vertices of a triangle to the Brocard-circle, θ the Brocard-angle, and Δ the area of the triangle; prove that (1)

$$t_1 = \frac{bc}{(a^2+b^2+c^2)^{\frac{1}{2}}},\quad t_2 = \frac{ca}{(a^2+b^2+c^2)^{\frac{1}{2}}},\quad t_3 = \frac{ab}{(a^2+b^2+c^2)^{\frac{1}{2}}};$$

and hence (2)

$$\frac{2\Delta}{t_1^2+t_2^2+t_3^2} = \sin 2\theta \text{ ..... } 27$$

8387. (H. MacColl, B.A.)—Let u and v be often taken at random within certain limits, and for each pair of these random values let the equations $f(x, y) = u$, and $\phi(x, y) = v$ determine the position of a point (or points) referred to Cartesian axes. These points will ultimately form some shaded geometrical figure, and the shading, depending on the functions f and ϕ, will not, generally speaking, be uniform. Find an expression for the relative depth of shading round any specified point x, y. 77

8388. (R. F. Davis, M.A.)—If ABC be a triangle, and AD, BE, CF perpendiculars upon the opposite sides, the lines joining the middle points of (AD, BC), (BE, CA), (CF, AB), each pass through the symmedian point. 97

8389. (Professor Sylvester, F.R.S.)—Prove that (1) if the roots of the equation $ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0$ are spaced out at harmonic intervals [*i.e.*, if p, q, r, s being the roots, $q-p, r-p, s-p$ are in harmonic progression] the cubic invariant $\begin{vmatrix} abc \\ bcd \\ cde \end{vmatrix}$ will vanish; (2) if the roots of the equations $Ax^2 + 2Bx + C = 0$, $A'x^2 + 2B'x + C' = 0$, taken alternately, are spaced out at harmonic intervals, the invariant $AC' - 2BB' + CA'$ will vanish; (3) the invariant $AC' - 2BB' + CA'$ is contained as a divisor in the cubic invariant of the biquadratic function

$$(Ax^2 + 2Bx + C)(A'x^2 + 2B'x + C');$$

and form (4) the value of the remaining factor or quotient in (3). ... 70

8390. (Professor Crofton, F.R.S.)—If m things are divided among n persons so that every one receives something, prove that (1) the number of ways this can be done is equal to the function $\Delta^n 0^m$ in the Calculus of finite differences, viz.,

$$\Delta^n 0^m = n^m - n(n-1)^m + \tfrac{1}{2}n(n-1)(n-2)^m - \&c.,$$

which vanishes when $m < n$. Find (2) the number of ways, if a particular set of r out of the n persons are all to receive something; also (3) if simply a number r, at least, are to do so; again (4), suppose all the things are to go to a number r, each one receiving something. 71

8393. (Professor Neuberg.)—Soient I, E, E′, E″ le centre du cercle inscrit et ceux des cercles ex-inscrits au triangle ABC, et soient F, F′, F″ les points où les bissectrices des angles BIC, CIA, AIB, rencontrent respectivement BC, CA, AB. Démontrer que les droites EF, E′F′, E″F″ se rencontrent en un même point L tel que les parallèles menées par L aux côtés du triangle EE′E″ et limitées respectivement par les angles du triangle ABC sont égales entre elles. 72

8394. (For Enunciation, see Question 8275.) 85

8396. (F. Purser, M.A.) — A variable conic circumscribes a given triangle, the normals at the vertices meeting in a point P. Show that the locus of P is a cubic passing through the vertices of the triangle, the circumcentre, orthocentre, incentre, and three excentres. The circumcentre is also the centre of the cubic, and the lines joining it to the middle points of the sides of the triangle are the three asymptotes. 105

8397. (Professor Hudson, M.A.)—If the orbits of two comets (supposed parabolic) intersect the orbit of the earth (supposed circular) in the same two given points, prove that, if t_1, t_2 be the times in which the comets respectively move from one of these points to the other, and a year be the unit of time, $(t_1+t_2)^{\frac{1}{2}}+(t_1-t_2)^{\frac{1}{2}}=(4/3\pi)^{\frac{1}{2}}$. 101

8398. (Professor Mathews, M.A.) — If i, w, v are defined by the equations $i^2+1=0$, $w^2+w+1=0$, $v=i+w$, prove that

$$w=-\frac{v^4+4v^3+8v^2+12v+3}{4v^3+6v^2+16v+7},\quad i=\frac{5v^4+10v^3+24v^2+19v+3}{4v^3+6v^2+16v+7}\ldots 72$$

8399. (The Editor.) — If from a point P in the base BC of a triangle ABC, PEF be drawn parallel to the bisector of the angle A, cutting AB, AC in E, F, and POQ be drawn through the in-centre O, cutting a parallel to BC through A in Q; prove that (1) AE=AF=AQ; and (2) if P be the point of contact of the in-circle with BC, and p the perpendicular from A on BC, then $PE.PF=p^2$. 87

8400. (Rev. J. J. Milne, M.A. Similar to the Editor's Question 8052.)—A point P is taken on the base of a triangle, and two lines including a given angle drawn through it so as to cut off from the triangle a quadrilateral of maximum area. Prove that (1), whatever be the position of P, the sides of the quadrilateral through P are equal to one another; and (2) in the position of P which gives the greatest maximum, those sides are equally inclined to the base of the triangle. 88

8404. (Maurice D'Ocagne.) — La courbe décrite par le centre de gravité des points de rencontre d'une conique fixe et d'un cercle mobile, de rayon nul, est transformée linéaire de la courbe décrite par le centre de ce cercle. 92

8407. (Captain H. Brocard.) — Les lignes qui joignent les sommets du triangle ABC aux points de Brocard ω et ω' déterminent sur les côtés opposés deux systèmes de trois points, qui sont les sommets de deux triangles, inégaux et dissemblables, mais équivalents. Trouver le rapport des surfaces de ces triangles à celle du triangle donné. 73

8409. (Émile Vigarié.) — On mène deux tangentes AB, AC à une parabole de foyer F et l'on circonscrit un cercle de centre O au triangle ABC. Démontrer que, si P est le pôle de BC par rapport à ce cercle: (1) la droite AP est médiane antiparallèle (symédiane) du triangle ABC;

(2) la droite AP passe par le foyer F de la parabole; (3) l'angle AFO est droit; (4) les points O, P, B, C sont sur un cercle de diamètre OP qui passe par le point F. 104

8417. (W. H. Blythe, M.A.)—Find whether the equation

$$\frac{d^2w}{dx^2} + \frac{2}{x}\frac{dw}{dx} + \left(\frac{2}{x^2} - n^2\right)w = 0 \text{ has any real solution}\ldots\ldots 78$$

8418. (W. J. C. Sharp, M.A.)—If the sides of a spherical triangle ABC be bisected in the points D, E, F, show that

$$\cos \mathrm{ED} \sec \tfrac{1}{2}c = \cos \mathrm{EF} \sec \tfrac{1}{2}a = \cos \mathrm{FD} \sec \tfrac{1}{2}b. \ldots\ldots 92$$

8425. (Arthur Hill Curtis, LL.D., D.Sc. Suggested by Quest. 8354.)—Inscribe in a given quadrilateral a quadrilateral similar to a given one. 120

8430. (R. Curtis, M.A.)—A and B are two fixed points on an ellipse, C a variable point, D the centre, DE, DF parallels to BC, BA; prove that the area of the sector DEF is constant..... 108

8432. (Professor Crofton, F.R.S.)—Prove that

$$\Delta^n 0^{n+r} = n! \,.\, \Sigma(n+1)\,\Sigma(n+2)\ldots\Sigma(n+r),$$

the integration being with respect to n. Thus

$$\Delta^n 0^{n+1} = \frac{n(n+1)}{2!}n!, \quad \Delta^n 0^{n+2} = \frac{n(n+1)(n+2)(3n+1)}{4!}n!, \text{ \&c.} \quad 86$$

8434. (Professor Neuberg.)—Sur une base donnée BC et d'un même côté, on construit six triangles semblables entre eux; démontrer que les sommets de ces triangles sont sur une même circonférence. 87

8435. (Professor Catalan.)—Démontrer la formule

$$\left\{\mathrm{L}\cdot\frac{-x+z+(1-2xz+z^2)^{\frac{1}{2}}}{1+x}\right\}\left\{\mathrm{L}\cdot\frac{-x+z+(1-2xz+z^2)^{\frac{1}{2}}}{1-x}\right\} - 2\sum_0^\infty \mathrm{A}_n\frac{z^{n+1}}{n+1}$$

dans laquelle $\mathrm{A}_0 = \frac{1}{2}\mathrm{L}\cdot\frac{1-x}{1+x}$, $\mathrm{A}_1 = 1 + \mathrm{A}_0 x$; et, à partir de $n = 1$,

$$(n+1)\mathrm{A}_{n+1} - (2n+1)x\mathrm{A}_n + n\mathrm{A}_{n-1} = 0. \ldots\ldots 114$$

8436. (Professor Sircom, M.A.)—A tricuspidal quartic passes through a fixed point, and has its cusps at fixed points; prove that the locus of its cuspidal centre is a conic touching the sides of the triangle of reference. 88

8437. (Professor Wolstenholme, M.A., Sc.D.)—Normals to a parabola ($y^2 = 4ax$) at P, Q, R meet in a point (X, Y); prove that the straight lines drawn from P, Q, R perpendicular to the opposite sides are normals to the parabola $y^2 = 16a(x + 12a)$, and that, if the corresponding tangents be drawn forming a triangle P′Q′R′, this triangle will be equal and similar to PQR, each of the three distances PP′, QQ′, RR′ being parallel to the axis and of length $10a + \mathrm{X}$. 124

8441. (Professor Hudson, M.A.)—If

$$x^2\cos\alpha\cos\beta + x(\sin\alpha + \sin\beta) + 1 = 0,$$

and

$$x^2\cos\beta\cos\gamma + x(\sin\beta + \sin\gamma) + 1 = 0,$$

prove that

$$x^2\cos\gamma\cos\alpha + x(\sin\gamma + \sin\alpha) + 1 = 0. \ldots\ldots 123$$

8444. (For Enunciation, see Question 8318.)..... 103

8446. (Captain H. Brocard.)—On construit (α) la parabole tangente à deux côtés du triangle et aux deux hauteurs correspondantes, (β) la parabole tangente aux pieds des bissectrices de l'angle compris entre ces deux côtés; démontrer (1) que ces deux coniques admettent pour foyer le pied de la hauteur partant du sommet de l'angle considéré; (2) déterminer leurs divers éléments et évaluer leurs paramètres; et (3) fixer la situation de leurs points d'intersection. 106

8448. (John Griffiths, M.A.)—If n be an odd prime number and s a number from 1 to $\frac{1}{2}(n-1)$, show that

$$(-)^{\frac{1}{2}(n-1)}\, n \sec n\theta = \sec\theta \left\{1 + 2\Sigma \frac{(-)^s \cos(s\pi/n)}{1-\sin^2(s\pi/n)\sec^2\theta}\right\}. \quad \ldots\ldots 109$$

8471. (Professor Neuberg.) — Trouver dans le plan d'un triangle ABC un point M tel que les perpendiculaires menées sur MA, MB, MC respectivement de A, B, C, ou de B, C, A, ou de C, A, B, concourent en un même point N ou N_1 ou N_2. Démontrer que les triangles ABC, NN_1N_2 sont équivalents et ont même centre de gravité..... 119

8478. (Professor Steggall, M.A.)—If l, m, n; l', m', n'; $\cos\alpha, \cos\beta, \cos\gamma$, be the direction cosines of the incident ray, the refracted ray, and the normal to the refracting surface at the point of incidence, prove that

$$\frac{l-\mu l'}{\cos\alpha} = \frac{m-\mu m'}{\cos\beta} = \frac{n-\mu n'}{\cos\gamma} \quad \ldots\ldots 102$$

8479. (Professor Hudson, M.A.)—If two straight lines A, B meet a quartic each in four points A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, and two parabolas be described through the points A_1, A_2, B_1, B_2, A_3, A_4, B_3, B_4; prove that the remaining points of intersection of the parabolas and the quartic lie on a conic section. 119

8480. (Professor Genese, M.A.) — ABCD is a quadrilateral, P, Q the intersections of opposite sides. Prove that the bisectors of opposite angles A and C, B and D, P and Q meet in points which lie in threes on six straight lines; prove also that, if A, B, C, D be concyclic, the bisectors of angles P and Q meet on the join of the mid-points of AC, BD, and divide the join in the ratio AC : BD. 111

8481. (The Editor.) — Through a given point O in the side AB of a triangle ABC draw a straight line POQ, cutting the side AC in P, and the prolongation of the side CB in Q, so that PC may have to QB the given ratio of $m : n$. 117

8485. (Capt. H. Brocard.)—On sait que la droite GZ, qui joint le centre de gravité au centre du cercle de Brocard, rencontre le cercle circonscrit au triangle en un point N situé également sur les lignes OD, KZ' et HD', et dont les coordonnées sont proportionnelles à $1/\cos(A+\theta)$, $1/\cos(B+\theta)$, $1/(C+\theta)$. Démontrer que le second point d'intersection V est sur la polaire du point G par rapport au cercle de Brocard, ou, en d'autres termes, que cette polaire passe par le point R diamétralement opposé à N sur le cercle circonscrit. Déterminer enfin les coordonnées du point V. 115

8495. (Rev. T. C. Simmons, M.A.)—Four passengers, A, B, C, D, entire strangers to each other, are travelling in a railway train, which contains l first-class, m second-class, and n third-class compartments. A and B are gentlemen whose respective *à priori* chances of travelling first, second, or third class are represented in each instance by λ, μ, ν; C and

D are ladies whose similar *à priori* chances are each represented by l, m, n. Prove, for all values of λ, μ, ν (except in the particular case when $\lambda : \mu : \nu = l : m : n$), that A and B are more likely to be found both in the company of the same lady than each with a different one..... 111

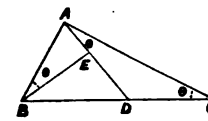
8496. (W. J. C. Sharp, M.A.)—If the number of partitions, without repetitions, of a number n be denoted by P_n, and the number with repetitions by Q_n; prove that

$$P_n + P_{n-2}Q_1 + P_{n-4}Q_2 \ldots\ldots = Q_n;$$

$$P_n + P_{n-1}Q_1 + P_{n-2}Q_2 + \ldots\ldots + Q_n = P_{2n} - P_{2n-1}Q_1 + P_{2n-2}Q_2 \ldots\ldots + Q_{2n};$$

$$P_{2n-1} - P_{2n-2}Q_1 + P_{2n-3}Q_2 - \ldots\ldots - Q_{2n-1} = 0 \ldots\ldots 123$$

8497. (R. Tucker, M.A.)—In a triangle ABC, $\angle ABC = 3ACB$: $\angle CAD = ABE = ACB = \theta$; show that the ratio of triangle BED to triangle ABC is $= \sin^2 5\theta \sin\theta : \sin^2 4\theta \sin 3\theta$, and that, when $\theta = 22\frac{1}{2}°$, 30°, this reduces to $1 : 2\sqrt{2}$ and 1 : 6 respectively.



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APPENDIX I.

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APPENDIX II.

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MATHEMATICS

FROM

THE EDUCATIONAL TIMES:

WITH ADDITIONAL PAPERS AND SOLUTIONS.

8216. (By Professor SYLVESTER, F.R.S.)—Let x, y, z, t be the distances of any point P, and x', y', z', t' of P′, from the faces of a fundamental pyramid divided respectively by the distances of the vertices a, b, c, d of the pyramid from those faces; prove that $PP'^2 = \Sigma (ab)^2 (x-x')(y'-y)$. [Of course a precisely similar formula applies to distances on a plane or in space of any number of dimensions.]

Solution by the PROPOSER.

It is obvious that PP'^2 is a homogeneous quadratic function of $x-x'$, $y-y'$, $z-z'$; but these three quantities are subject to a linear homogeneous equation, so that the square of any one of them is a linear a function of its products into each of the two others. Hence $PP'^2 = \Sigma m (x-x')(y-y')$. On making PP′ coincide with a, b respectively, it will be seen that $(ab)^2 = -m$, which proves the formula.

8224. (By S. H. BOULT, M.A.)—Express the Compound Interest on a sum of money for any number of years at any rate per cent. by a series.

Solution by D. BIDDLE; *the* PROPOSER; *and others.*

Let A = the original sum in pounds, b = rate of interest divided by 100, and n = number of years; then the amounts at end of 1st, 2nd, 3rd years are

$$A + Ab, \quad A + Ab + (A + Ab)\, b,$$

$$A + Ab + (A + Ab)\, b + \{A + Ab + (A + Ab)\, b\}\, b,$$

and so on, all of which are clearly resolvable into the series

$$A\left\{1 + nb + \frac{n(n-1)}{1.2} b^2 + \frac{n(n-1)(n-2)}{1.2.3} b^3 + \ldots + nb^{n-1} + b^n\right\}.$$

8172. (By JOHN GRIFFITHS, M.A.)—If n be an integer, show that

$$\mathrm{dn}\,\frac{\mathrm{K}}{2n}\,\mathrm{dn}\,\frac{3\mathrm{K}}{2n}\,\mathrm{dn}\,\frac{5\mathrm{K}}{2n} = \mathrm{dn}\,\frac{2n-1}{2n}\,\mathrm{K} = (k')^{\frac{1}{2}n};$$

k' being the modulus complementary to k.

Solution by Professor MATHEWS, M.A.

In general, $\mathrm{dn}\,u \,.\, \mathrm{dn}\,(\mathrm{K}-u) = k'$.

Writing successively $u = \frac{\mathrm{K}}{2n}, \frac{3\mathrm{K}}{2n}, \ldots\ldots \frac{(2n-1)\,\mathrm{K}}{2n}$,

and multiplying the results together, we get

$$\left\{\mathrm{dn}\,\frac{\mathrm{K}}{2n}\,.\,\mathrm{dn}\,\frac{3\mathrm{K}}{2n}\ldots\ldots\,\mathrm{dn}\,\frac{(2n-1)\,\mathrm{K}}{2n}\right\}^2 = k'^n;$$

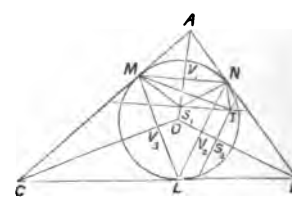
and, extracting the square root, $\mathrm{dn}\,\frac{\mathrm{K}}{2n}\,.\,\mathrm{dn}\,\frac{3\mathrm{K}}{2n}\ldots\ldots\,\mathrm{dn}\,\frac{(2n-1)\,\mathrm{K}}{2n} = k'^{\frac{1}{2}n}$.

[The positive sign has to be taken on the right, because every factor on the left hand is positive.]

8245. (By Professor NEUBERG.)—Trouver dans le plan d'un triangle ABC un point M tel que les perpendiculaires menées par M sur les bissectrices des angles A, B, C et limitées par ces angles soient égales entre elles. Trouver la longueur commune de ces droites.

Solutions by (1, 2) A. H. CURTIS, LL.D., D.Sc.; (3) *the* PROPOSER.

1. For greater generality, instead of supposing the intercepts mentioned in the question to be equal, let them be in the given ratios $1 : k : k'$, and be denoted by l_1, l_2, l_3. Adopting the notation proposed at page 211, let the point required be denoted by I, instead of M as in the question; then, if AO, BO, CO, the bisectors of the angles of the triangle, cut MN, &c., in points V_1, V_2, V_3, and the required lines in S_1, S_2, S_3, $l_1/\mathrm{MN} = \mathrm{AS}_1/\mathrm{AV}_1$; therefore $l_1 \times \mathrm{AV}_1 = \mathrm{AS}_1 \times \mathrm{MN} = 2$ area of quadrilateral AMIN, or, denoting $\mathrm{AV}_1, \mathrm{BV}_2, \mathrm{CV}_3$ by $\lambda_1, \lambda_2, \lambda_3$, we have $\lambda_1 l_1 + \lambda_2 l_2 + \lambda_3 l_3 = 2\Delta\mathrm{ABC}$ (1), or $l_1(\lambda_1 + k\lambda_2 + k'\lambda_3) = 2\Delta\mathrm{ABC}$, therefore l_1 is known in magnitude, also $l_2 \equiv kl_1$, and $l_3 \equiv k'l_1$ are also known. Draw then l_1 and l_2 of these lengths, parallel to MN and NL, and terminated by the sides of the triangle; their intersection will give I, and the line through I parallel to LM must $= l_3$, by the relation (1).

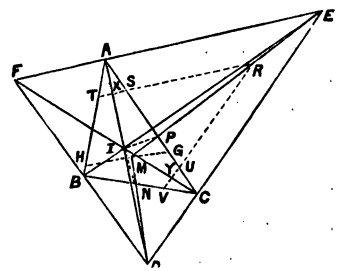


In the case proposed $k = k' = 1$, and

$$l_1 = l_2 = l_3 = \frac{2\Delta ABC}{\lambda_1 + \lambda_2 + \lambda_3} = \frac{2\Delta ABC}{AV_1 + BV_2 + CV_3}.$$

2. Otherwise thus:—The trilinear coordinates of I may be found by the conditions $\beta + \gamma = l_1 \cos \frac{1}{2}A$, $\gamma + \alpha = l_2 \cos \frac{1}{2}B$, $\alpha + \beta = l_3 \cos \frac{1}{2}C$, which are geometrically obvious, to be $\alpha = l_2 \cos \frac{1}{2}B + l_3 \cos \frac{1}{2}C - l_1 \cos \frac{1}{2}A$, &c., or $\alpha = l_1 (k \cos \frac{1}{2}B + k' \cos \frac{1}{2}C - \cos \frac{1}{2}A)$, &c., while the values above deduced for l_1, l_2, l_3 are easily obtained from the relation $2\Delta ABC = a\alpha + b\beta + c\gamma$.

3. Or again, thus:—Soient EDF le triangle formé par les bissectrices intérieures, et I le centre du cercle inscrit. Cherchons le lieu des points R, tels que les angles BAC, ACB interceptent sur des parallèles à EF, ED, menées par R, des segments égaux ST et UV.



Or $TS = 2XS = 2AX \cdot \tan \frac{1}{2}A$,

$VU = 2YU = 2CY \tan \frac{1}{2}C$;

par suite

$$AX \tan \tfrac{1}{2}A = CY \tan \tfrac{1}{2}C,$$

et le lieu de R est une droite passant par E. Le point E correspond à $TS = UV = 0$; on obtient un point du lieu en prenant le pied P de la bissectrice de l'angle AIC.

Donc le point cherché M est à l'intersection des droites joignant les centres des cercles en-inscrits à ABC, aux pieds des bissectrices des angles BIC, AIC, AIB.

Remarque—Les distances de M sur côtés du ΔDEF sont proportionnelles à $\cot \frac{1}{2}A$, $\cot \frac{1}{2}B$, $\cot \frac{1}{2}C$.

Soient α, β, γ les distances de M aux côtés de ABC. Si GH est perpendiculaire à AI, on a $HG = MG + MH = \frac{\beta}{\cos \frac{1}{2}A} + \frac{\gamma}{\cos \frac{1}{2}A}$. Donc, si l est la longueur HG, on a

$$\beta + \gamma = l \cos \tfrac{1}{2}A;\ \gamma + \alpha = l \cos \tfrac{1}{2}B,\ \alpha + \beta = l \cos \tfrac{1}{2}C, \quad \ldots\ldots\ldots\ldots(1),$$

$$a\alpha + b\beta + c\gamma = 2S;$$

d'où, en faisant la somme des équations multipliées par $b + c - a$, $a + c - b$, $a + b - c$,

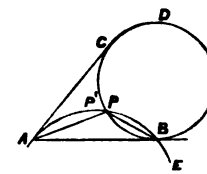
$$l = \frac{2S}{(p-a)\cos \frac{1}{2}A + (p-b)\cos \frac{1}{2}B + (p-c)\cos \frac{1}{2}C}.$$

8168. (By S. Roberts, M.A.)—Let A, B be two given points. A series of circles touch the straight line joining A, B at B. Another series of circles pass through A and B. The locus of the variable intersection of a circle of one series with a circle of the other, when these have AC as a common tangent, is the inverse of a parabola whose axis is AB with respect to B as pole. Show this; and by means of the result determine the regions

within which a point P′ must lie so that a triangle with a base, given in position and magnitude, may have the point P′, (1) for a BROCARD-point of either species; (2) of one species.

Solution by the PROPOSER.

1. Let ABE, CBD be a pair of circles fulfilling the conditions and intersecting again in P; and write ϕ, θ for the angles BAC, PBA respectively; k, ρ for AB and PB respectively; and r_1, r_2 for the radii of ABE, BCD respectively. Then we



have $k = 2r_1 \sin\phi$, $r_2 = k \tan \frac{1}{2}\phi$,

$$\rho = 2r_2 \sin\theta = 2r_1 \sin \mathrm{PAB} = 2r_1 \sin(\phi - \theta)$$

$$= 2k \tan\tfrac{1}{2}\phi \sin\theta = k \frac{\sin(\phi-\theta)}{\sin\phi},$$

or
$$\frac{k \sin\theta}{\rho - k\cos\theta} = -\tan\phi = \frac{4\rho k \sin\theta}{\rho^2 - 4k^2 \sin^2\theta},$$

or $4\rho(\rho - k\cos\theta) - (\rho^2 - 4k^2\sin^2\theta) = 0 = 4(\rho^2 + k^2) - (\rho + 2k\cos\theta)^2$.

Putting ρ' for ρ^{-1} in this equation, we have $4(1 + k^2\rho'^2) - (1 + 2kx)^2 = 0$, a parabola.

2. Now, if P′ is a Brocard-point and P′BA the Brocard-angle, and we draw a circle touching AB at B and passing through P′, AC must cut this circle and the two intersections are the vertices of two triangles to which the Brocard-point P′ may belong. The point P′ must consequently lie within the parabola-inverse described by P, B being the cusp.

If the Brocard-point is of the other kind, we have a corresponding construction relative to the extremity A.

The area common to the two parabola-inverses solves the case (1), the areas not common to both solve the case (2). It is easy to determine certain limits. Thus the real intersections of the parabola-inverses are the centres of gravity of the equilateral triangles on the base AB.

8307. (By Professor CATALAN.)—Démontrer que

$$\cos^n x + \cos^{n-1} x \cos x + \cos^{n-2} x \cos 2x + \ldots + \cos nx = \sin(n+1)x \operatorname{cosec} x.$$

Solution by Professor IGNACIO BEYEN, M.A.; H. LOUDON; *and others.*

Let $C = \cos^n x + \cos^{n-1} x \cos x + \ldots + \cos nx$,

$S = \cos^{n-1} x \sin x + \ldots + \sin nx$,

therefore $C + Si = \cos^n x + \cos^{n-1} x \cdot e^{xi} + \ldots + e^{nxi}$

$$= \frac{e^{(n+1)xi} - \cos^{n+1} x}{e^{xi} - \cos x} = \frac{\cos(n+1)x + i\sin(n+1)x - \cos^{n+1} x}{i \sin x},$$

therefore $iC \sin x - S \sin x = \cos(n+1)x - \cos^{n+1} x + i \sin(n+1)x$,

therefore, equating unreal parts, $C = \dfrac{\sin(n+1)x}{\sin x}$.

[From this theorem it follows that

$$1 - \frac{n-1}{1}\sin^2 x\cos^2 x + \frac{(n-2)(n-3)}{1\,.\,2}5a^4 x\cos^4 x - \frac{(n-3)(n-4)(n-5)}{1\,.\,2\,.\,3}\sin^6 x\cos^6 x + \ldots$$
$$= \sin^{2n} x + \sin^{2n-2} x\cos^2 x + \sin^{2n-4} x\cos^4 x + \ldots + \cos^{2n} x.]$$

Analytic Investigation of Formulæ for Radii of Curvature, &c.

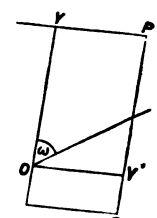
By E. W. Symons, M.A.

Let $x\cos\omega + y\sin\omega = p$...(1) be the equation of the tangent at any point to a given curve. If we regard p as a function of ω and differentiate the equation with respect to ω, we have the equation of another straight line

$$-x\sin\omega + y\cos\omega = \frac{dp}{d\omega} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

Now, by the theory of envelopes, (1) and (2) intersect on the given curve, and because (2) is perpendicular to (1), it must be the equation of the normal to the envelope of (1), or, in other words, (2) touches the evolute, and similarly, if we differentiate (2), we get the equation of a tangent to the second evolute, and so on. Hence we have the following result, $dp/d\omega$, $d^2p/d\omega^2$.....$d^np/d\omega^n$ are respectively the perpendiculars from the origin on the tangents to the first, second.....nth evolutes (α).

Again, if P be a point on the curve C, the corresponding point on evolute, the line through C parallel to PY is a tangent to second evolute. Hence, by (α),



$$Cy' = \frac{d^2p}{d\omega^2}, \text{ and } CP = \rho, \quad Py' = p;$$

therefore $$\rho = Cy' + Py' = \frac{d^2p}{d\omega^2} + p = \frac{ds}{d\omega} \quad \ldots\ldots\ldots\ldots \quad (\beta)$$

Now, by (2), formulæ for the evolute may be obtained from those of the given curve by changing p into $dp/d\omega$, and ω into $\frac{1}{2}\pi + \omega$. Hence, if ρ' be radius of curvature for evolute, and s' the corresponding arc on it,

$$\rho' = \frac{ds'}{d\omega} = \frac{dp}{d\omega} + \frac{d^3p}{d\omega^3} = \text{by } (\beta)\ \frac{d\rho}{d\omega};$$

whence $ds' = d\rho$, and therefore $s' = \rho_1 - \rho_2$. Again, squaring and adding equations (1) and (2), we get $x^2 + y^2 = r^2 = p^2 + \left(\frac{dp}{d\omega}\right)^2$,

therefore, by differentiation, $$r\frac{dr}{dp} = p + \frac{d^2p}{d\omega^2} = \rho, \text{ by } (\beta).$$

8000. (By Professor WOLSTENHOLME, M.A., Sc.D.)—In a cubic with three real asymptotes and an acnode, prove that the area between any branch of the curve, and the two asymptotes which include it, is one-third of the area of the triangle formed by the asymptotes. [The equations of the curve in Cartesian and Areal coordinates are

$$27xy\left(1-\frac{x}{a}-\frac{y}{b}\right)=ab,\ 27xyz=(x+y+z)^3.]$$

Solution by D. EDWARDES.

Let O be the origin, OHK the triangle formed by the asymptotes. The line QR in the figure is a tangent parallel to OH, and $OR=\frac{4}{3}a$. The equation solved for y is $y=\frac{b}{2a}\left[a-x\pm\frac{(x-\frac{1}{3}a)(x-\frac{4}{3}a)^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right]$,

whence we see that AB = CD. Hence the area between the curve, the x axis, the asymptote HK and the tangent QR is

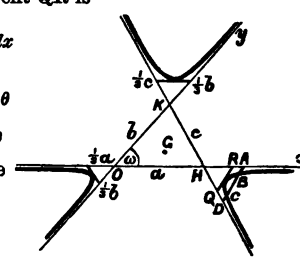
$$\frac{b}{a}\sin\omega\int_{\frac{4}{3}a}^{\infty}\left[x-a-\frac{(x-\frac{1}{3}a)(x-\frac{4}{3}a)^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right]dx$$

$$=\frac{b}{a}\sin\omega\int_0^{\frac{1}{3}\pi}\left[(\tfrac{4}{3}a\sec^2\theta-a)\tfrac{8}{3}a\sin\theta\sec^3\theta\right.$$
$$\left.-\tfrac{8}{9}a^2(4\sec^2\theta-1)\sin^2\theta\sec^3\theta\right]d\theta$$

(where $x=\frac{4}{3}a\sec^2\theta$, and ω is the angle between the asymptotes)

$$=\tfrac{4}{9}ab\sin\omega\left[\frac{3\sin^2\theta-2\sin^3\theta-1}{\cos^4\theta}\right]_0^{\frac{1}{3}\pi}$$

$$=\tfrac{1}{9}ab\sin\omega,$$



and area of triangle $HQR=\frac{1}{3}\cdot\frac{1}{3}a\cdot\frac{1}{3}b\sin\omega=\frac{1}{18}ab\sin\omega$. Therefore, area in question $=\frac{1}{6}ab\sin\omega=\frac{1}{3}$ (area of triangle OHK). Similarly for the other branches. There is an acnode at the centroid of the triangle formed by the asymptotes.

FINAL NOTE ON QUESTION 7695. *By* Rev. T. C. SIMMONS, M.A.

Since the publication of the remarks on p. 87 of Vol. XLIII., I have been agreeably surprised to find that the above Question was fully discussed of old, independently, by HUYGHENS and JAMES BERNOULLI. It is reassuring to see one's own conclusions corroborated by such high authorities as these; and readers who have carefully considered the Question will not need to be told that neither HUYGHENS nor JAMES BERNOULLI agree with Mr. DODGSON (who asserts that the resulting ratio $\frac{30}{31}$ is only *approximative.*) In fact, it is clear that as A's chance of winning at the first throw is *exactly* $\frac{5}{36}$, at the third throw *exactly* $\frac{5}{36}\cdot\frac{155}{216}$, &c., his whole chance is *exactly* represented by $\frac{5}{36}[1+\frac{155}{216}+(\frac{155}{216})^2+\ldots]$. And as B's

whole chance is similarly represented *exactly* by $\frac{31}{36} \cdot \frac{6}{36}[1 + \frac{1}{31}\frac{5}{16} + \ldots]$, the ratio of A's chance to B's chance must be *exactly* the ratio of $\frac{5}{36}$ to $\frac{31}{36} \cdot \frac{6}{36}$, or *exactly* the ratio of 30 to 31 ; a conclusion entirely independent of all considerations as to "whether an infinite converging series does or does not reach its limit." (See also the two solutions given in TODHUNTER's *History of Probability*, pp. 24 and 60).

As to Mr. DODGSON's remarks on p. 27 of Vol. XLIV., I disagree with them *in toto*. The first paragraph is irrelevant; in the fourth, axioms are confused with definitions; the third alone includes the real pith of the matter. But what extraordinary statements! How *can* a point which falls at C be reckoned as not falling in BC? If, for instance, AB be bisected at C, is it allowable to reckon C as belonging to AC and not to BC? Mr. DODGSON is evidently in sore straits, as are all logicians who try to prove conclusions inconsistent with their own definitions. A point being known to lie on one or other of two separate equal lines, its chance of lying on either line is of course *exactly* $\frac{1}{2}$. But, if the lines are brought together so as to become conterminous, Mr. DODGSON asserts that its chance of lying on either line is now *infinitesimally less than* $\frac{1}{2}$; his point of view apparently being this, that two points, formerly separate, have coalesced, thereby rendering the length of the united line infinitesimally less than was the sum of the two separate lines! "Really" (to use Mr. DODGSON's own words), "one is almost ashamed of having to refute so transparent a fallacy."

With reference to the word infinitesimal, I am unable myself to conceive an infinitesimal quantity which has not, corresponding to it, an infinitesimal of a higher order. And, unless I am mistaken, a quantity than which nothing can be conceived smaller, is indistinguishable from absolute zero.

It only remains to add that Mr. KNOWLES has wrongly imputed to me the statement that $\cdot\dot{9}$ is absolutely equal to 1. Whatever my *opinion* may be (and it is of no importance), I carefully refrained from making any such statement, my criticism having been at first confined to the inaccurate use of the word *approximate*. According to the best usage (that of TODHUNTER for instance), when two quantities are stated to be approximately equal, it is meant that their difference is small compared with either, but *finite*. But Mr. DODGSON's subsequent silence on this word may be construed into an admission that on p. 86 of Vol. XLIII. he used it incorrectly.

8384. (By W. J. MCCLELLAND, M.A.)—If t_1, t_2, t_3 be the lengths of the tangents from the vertices of a triangle to the Brocard-circle, θ the Brocard-angle, and Δ the area of the triangle; prove that (1)

$$t_1 = \frac{bc}{(a^2+b^2+c^2)^{\frac{1}{2}}}, \quad t_2 = \frac{ca}{(a^2+b^2+c^2)^{\frac{1}{2}}}, \quad t_3 = \frac{ab}{(a^2+b^2+c^2)^{\frac{1}{2}}};$$

and hence (2)
$$\frac{2\Delta}{t_1^2+t_2^2+t_3^2} = \sin 2\theta.$$

Solution by Captain H. BROCARD.

1. Z étant le milieu du diamètre du cercle OK, l'on a les relations
$(OA)^2 + (AK)^2 = 2(AZ)^2 + 2(OZ)^2$, $t_1^2 = (AZ)^2 - (OZ)^2$, $OA = R$,

$OZ = \frac{2R(p^4-n^4)^{\frac{1}{2}}}{m^2}$, $\tan(KAC) = \frac{c \sin A}{b + c \cos A}$, avec $m^2 = \Sigma a^2$, $n^4 = \Sigma a^2 b^2$, $p^4 = \Sigma a^4$. Mais le point K a pour coordonnées trilinéaires $\frac{2\Delta a}{m^2}$, etc., on en déduit

$$AK = \frac{bc(2b^2+2c^2-a^2)^{\frac{1}{2}}}{m^2} = \frac{2m_1 bc}{m^2},$$

m_1 étant la médiane issue du point A. On en conclut

$$t_1^2 = \frac{4m_1^2 b^2 c^2 + 3R^2(2n^4-p^4)}{2m^4} = \frac{2m^2 b^2 c^2 + 16\Delta^2 . 3R^2 - 3a^2 b^2 c^2}{2m^4} = \frac{b^2 c^2}{m^2}.$$

2. On a $\Sigma t_1^2 = \frac{n^4}{m^2}$. Mais $\tan\theta = \frac{4\Delta}{m^2}$, $\cos\theta = \frac{m^2}{2n^2}$, $\sin\theta = \frac{2\Delta}{n^2}$, et $\sin 2\theta = \frac{2\Delta m^2}{n^4}$, ou $\sin 2\theta = \frac{2\Delta}{\Sigma t_1^2}$.

[If Ω be one of the Brocard-points, and B_1 the vertex of the triangle drawn on the side AC of the given triangle and having each of its base angles equal to θ, we have at once $A\Omega \sin A = c \sin\theta$, $AB_1 \sin 2\theta = b \sin\theta$; hence, on multiplying, B_1 being a point on the Brocard-circle, we have

$$t_1^2 = bc \sin\theta \,/\, 2 \sin A \cos\theta = b^2 c^2 \,/\, a^2 + b^2 + c^2.]$$

8250. (By Professor MACFARLANE, F.R.S.E.)—Prove that

$$\frac{1}{2} + \frac{1}{2.4} + \frac{1.3}{2.4.6} + \ldots \text{ad inf.} = 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1);$$

$$\frac{1}{2} - \frac{1.3}{2.4.6} + \frac{1.3.5.7}{2.4.6.8.10} - \ldots \text{ad inf.} = \left\{\frac{(2^{\frac{1}{2}}-1)}{2}\right\}^{\frac{1}{2}} \ldots\ldots\ldots (2).$$

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.

1. If S_1, S_2 denote the two series, we have, identically,

$$(1-x^2)^{-\frac{1}{2}} = 1 + \tfrac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \&c.$$

Multiplying each side by x and integrating, we obtain

$$C - (1-x^2)^{\frac{1}{2}} = \tfrac{1}{2}x^2 + \frac{x^4}{2.4} + \frac{1.3x^6}{2.4.6} + \&c.,$$

but, as the right-hand member vanishes for $x = 0$, $C = 1$; therefore, changing x^2 into y, $1 - (1-y)^{\frac{1}{2}} = \tfrac{1}{2}y + \frac{y^2}{2.4} + \frac{1.3y^3}{2.4.6} + \&c.$,

or

$$\frac{1-(1-y)^{\frac{1}{2}}}{y} = \frac{1}{2} + \frac{y}{2.4} + \frac{1.3y^2}{2.4.6} + \&c. \quad \ldots\ldots\ldots\ldots (a).$$

Let $y = 1$, therefore $1 = \frac{1}{2} + \frac{1}{2.4} + \frac{1.3}{2.4.6} + \&c.$, or $S_1 = 1$.

2. Again, from (a), $\dfrac{(1+y)^{\frac{1}{2}}-1}{y} = \frac{1}{2} - \dfrac{y}{2.4} + \dfrac{1.3y^2}{2.4.6} - \&c.$ .....(b);

adding (a) and (b), we obtain

$$\frac{(1+y)^{\frac{1}{2}}-(1-y)^{\frac{1}{2}}}{y} = 2\left\{\frac{1}{2} + \frac{1.3y^2}{2.4.6} + \frac{1.3.5.7y^4}{2.4.6.8.10} + \&c.\right\},$$

or,
$$\left\{\frac{2-2(1-y^2)^{\frac{1}{2}}}{y^2}\right\}^{\frac{1}{2}} = 2\left\{\frac{1}{2} + \frac{1.3y^2}{2.4.6} + \frac{1.3.5.7y^4}{2.4.6.8.10} + \&c.\right\}.$$

Let $y^2 = -1$, therefore

$$\left\{2\left(2^{\frac{1}{2}}-1\right)\right\}^{\frac{1}{2}} = 2\left\{\frac{1}{2} - \frac{1.3}{2.4.6} + \frac{1.3.5.7}{2.4.6.8.10} - \&c.\right\} = 2S_2.$$

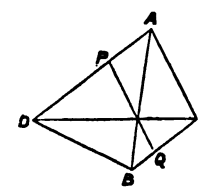
8156. (By Professor Sylvester, F.R.S.)—If two non-intersecting edges of a pyramid and the cosine of their inclination be multiplied together, prove that one of the three products thus formed is equal to the sum of the other two. Show also that the shortest distance between any given pair of non-intersecting edges of a pyramid is equal to the quotient of its content by the square root of a quadratic function of its six squared edges.

Solution by Professor Mathews, M.A.

Let the vectors DA, DB, DC be represented by α, β, γ respectively; then

$$S\alpha(\beta-\gamma) + S\beta(\gamma-\alpha) + S\gamma(\alpha-\beta) \equiv 0;$$

and the geometrical interpretation of this is the first part of the proposition.



Let PQ be the shortest distance between DA, BC; and suppose $S\pi\rho = 1$ to be the equation of the plane through BC parallel to DA; then

$$S\pi\beta = 1, \quad S\pi\gamma = 1, \quad S\pi\alpha = 0,$$

whence we find
$$\pi = \frac{V\gamma\alpha + V\alpha\beta}{S\alpha\beta\gamma} = \frac{V\alpha(\beta-\gamma)}{S\alpha\beta\gamma},$$

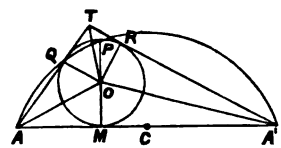
and
$$PQ = \frac{1}{T.\pi} = \frac{S\alpha\beta\gamma}{T.V\alpha(\beta-\gamma)}.$$

Now, $S\alpha\beta\gamma$ = six times volume of tetrahedron, and $T.V\alpha(\beta-\gamma)$ can be expressed as the square root of a rational function of the lengths of the edges.

8165. (By the Editor.)—If from the ends (A, A′) of the major axis AA′ of an ellipse, tangents (AQT, A′RT) are drawn to a circle on an ordinate PM to this axis as diameter, prove that the distance of the point (T) of intersection of these tangents from either of their points (Q, R) of contact is constant, and has to the axis of the ellipse the ratio $1-e^2 : 3+e^2$, where e is the eccentricity of the ellipse.

Solutions by (1) Rev. T. Galliers, M.A.; R. Knowles, B.A.; *and others*; (2) D. Edwardes; Prof. Chakravarti, M.A.; *and others.*

1. The circle in question will be the in-circle of the triangle TAA'. Let the angles of this triangle be denoted by T, A, A' respectively, and let r = radius of circle; then



$$TQ = r\cot\tfrac{1}{2}T = r\tan\tfrac{1}{2}(A+A')$$

$$= r\,\frac{\cot\frac{1}{2}A+\cot\frac{1}{2}A'}{\cot\frac{1}{2}A\cot\frac{1}{2}A'-1} = \frac{AM+A'M}{(AM\,.\,A'M)/r^2-1} = \frac{2a}{(a^2/b^2)(PM^2/r^2)-1}$$

$$= \frac{2a}{(4a^2/b^2)-1} = \frac{2ab^2}{4a^2-b^2} = 2a\left(\frac{1-e^2}{3+e^2}\right).$$

2. If p be the perpendicular from the vertex A upon the base BC of a triangle, r the in-radius, Δ the area of the triangle, and s the semi-perimeter, then $\frac{2r}{p} = 1-\tan\frac{1}{2}B\tan\frac{1}{2}C$, and $rs = \Delta = \frac{1}{2}pa$, whence $s = \frac{a(\frac{1}{4}a^2-x^2)}{\frac{1}{4}a^2-x^2-y^2}$, where (x, y) are the coordinates of the in-centre referred to the mid-point of the base as origin, the base being the axis of x. But, by the conditions of the question, we have $\frac{4x^2}{a^2}+\frac{4y^2}{b^2} = 1$, whence $s = \frac{4a}{3+e^2}$, and is therefore constant. Also $TQ = TR = t = s-a$, therefore $t : a = 1-e^2 : 3+e^2$.

7624. (By Nilkanta Sarkar, B.A.)—If a train, consisting of p carriages, each of which will hold q men, contains $pq-m$ men, find the chance that another man getting in, and being equally likely to take any vacant place, will travel in the same carriage with a given passenger.

Solutions by (1) D. Biddle; (2) *the* Rev. T. C. Simmons, M.A.

1. Let A denote the particular passenger seated, B the new-comer. Then, the chance of B travelling in the same carriage with A, depends upon the number of vacant seats in A's carriage, as compared with those in the whole train. This we cannot find for any single instance: we must therefore suppose it to vary from zero to m, or to $(q-1)$ if the latter be less than m, and we must find the comparative frequency with which, under the conditions, the several numbers would occur.

Let c_0, c_1, c_2, c_3, &c. = the several chances of 0, 1, 2, 3, &c., vacant seats being in A's carriage. Then

$$P = \frac{0}{m}c_0+\frac{1}{m}c_1+\frac{2}{m}c_2+\frac{3}{m}c_3+\ldots+\frac{m}{m}c_m$$

$$\left(\text{or } \frac{q-1}{m}c_{q-1} \text{ as the case may be}\right).$$

The total number of ways in which $(q-1)$ things can be taken out of $(pq-1)$ things is $\frac{(pq-1)!}{(pq-q)!\,(q-1)!}$, where the arrangement is indifferent; and the ways in which 1, 2, 3, &c. things out of m things, together with $(q-2)$, $(q-3)$, $(q-4)$, &c. things out of $(pq-m-1)$ things, can be taken, follow the same rule.

Let $\frac{(pq-q)!\,(pq-m-1)!}{(pq-1)!} = k$, a constant factor, and let $(pq-q-m) = f$,

then $$c_0 = \frac{k}{f!}, \quad c_1 = \frac{(q-1)\,m}{f+1}c_0, \quad c_2 = \frac{(q-2)\,(m-1)}{2\,(f+2)}c_1,$$

$$c_3 = \frac{(q-3)\,(m-2)}{3\,(f+3)}c_2, \quad c_4 = \frac{(q-4)(m-3)}{4\,(f+4)}c_3, \text{ and so on.}$$

It is then easy to find P by the above formula.

Thus, let $p = 20$, $q = 10$, $m = 4$, then

$$c_0 = \frac{190\,.\,189\,.\,188\,.\,187}{199\,.\,198\,.\,197\,.\,196} = \frac{1,262,451,960}{1,521,390,024}.$$ Consequently,

$$\frac{1}{m}c_1 = \frac{9}{187}c_0 = \frac{62,759,720}{1,521,390,024}, \quad \frac{2}{m}c_2 = \frac{24}{188}\cdot\frac{1}{m}c_1 = \frac{7,756,560}{1,521,390,024}$$

$$\frac{3}{m}c_3 = \frac{7}{189}\cdot\frac{2}{m}c_2 = \frac{287,280}{1,521,390,024}, \quad \frac{4}{m}c_4 = \frac{2}{190}\cdot\frac{3}{m}c_3 = \frac{3,024}{1,521,390,024},$$

and $$P = \frac{62,759,720 + 7,756,560 + 287,280 + 3,024}{1,521,390,024}$$

$$= \frac{70,806,584}{1,521,390.024} = \cdot 04654 \text{ nearly, or about } \frac{2}{43}.$$

2. *Otherwise:*—The above method (which I have had the opportunity of seeing) is not easily understood, nor its application easy to particular cases; moreover, I believe the result to be incorrect. We are, of course, to assume (what is never actually the case) that the passengers, unenticed by the charms of corner-seats, agreeable neighbours, or emptier compartments, all seat themselves *at random*. This being premised, and A being supposed already seated, the chance that any subsequent passenger will occupy any particular seat is clearly the same whether he enter the train first or last of the remaining $pq-m$ men. But, if B had entered next after A, his chance of sitting in A's seat would have been $(q-1)/(pq-1)$. This therefore, which is independent of m, I believe to be the true formula required.

Taking $p = 20$, $q = 10$, $m = 4$, we obtain $P = \frac{9}{189} = \cdot 04522\ldots$, which is slightly less than Mr. Biddle's result.

8234. (By Rev. J. J. Milne, M.A.)—If the normal PG of a conic be given in magnitude and position, and also the eccentricity, prove geometrically that the *latus rectum* will envelope a conic of the same species.

Solution by R. J. Brevor, B.A.; G. G. Storr, B.A.; *and others.*

If S be a focus of the conic, whose normal is given, the locus of S (when e is not = 1) is the circle on MN as diameter, where M, N are the points dividing PG internally and externally in the ratio $1 : e$; and the envelope of the *latus rectum* is the conic having G for focus and this circle as auxiliary circle.

The focus G will lie inside or outside MN, and the envelope will therefore be an ellipse or hyperbola according as e is $<$ or >1, *i.e.*, according as the original conic is an ellipse or hyperbola.

If the given eccentricity = 1, the locus of S is a straight line, and the envelope is a parabola having G for focus, and this straight line as tangent at the vertex.

7883. (By J. O'Regan.)—A and B play for a stake which is to be won by him who makes the highest score in 4 throws of a die. After two throws, A has scored 12, and B 9; find A's chance of winning.

Solution by Rev. T. C. Simmons, M.A.; N. Sarkar, B.A.; *and others.*

All we have to do is to find the chance of B's total not exceeding A's by so much as 3 in the remaining two throws. Now, the possible numbers obtainable by either A or B are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12; and they can occur respectively in 1, 2, 3, 4, 5, 6, 5, 4, 3, 2, 1 ways, as can be seen from the coefficients of consecutive powers of x in the expansion of $(x+x^2+x^3+x^4+x^5+x^6)^2$. Hence the number of ways in which B's total from the two throws can be at least 4 in excess of A's

$$= 1(5+6+5+4+3+2+1)+2(6+5+4+3+2+1)+3(5+4+3+2+1)$$
$$+4(4+3+2+1)+5(3+2+1)+6(2+1)+5.1$$
$$= 26+42+45+40+30+18+5 = 206;$$

the number of ways in which B's total can be exactly 3 in excess of A's

$$= 1.4+2.5+3.6+4.5+5.4+6.3+5.2+4.1 = 104;$$

and the total possible number of throws $= 6^4 = 1296$; hence B's chance of winning $= \frac{206}{1296}$, and the chance of an equality $= \frac{104}{1296}$; therefore A.'s chance $= \frac{986}{1296}$.

8264. (By S. Tebay, B.A.)—The shaft of a mine, 132 yards deep, is worked by two concentric grooved pulleys, each 9 feet diameter, revolving in opposite directions, and carrying a coiled rope $1\frac{1}{2}$ inches in diameter; find in what part of the shaft the cages pass each other.

Solution by the Proposer.

Let d be the depth of the mine, R the radius of each pulley, and r the radius of the rope. Then the radii of the several coils of rope are $R+r$, $R+3r$, $R+5r$, ... $R+(2n-1)r$, n being the number of coils. The sum of these is $Rn+n^2r$. Therefore

$$2\pi n(R+nr) = d \quad \text{.....(1).}$$

Hence in x turns the lower cage rises $2\pi x\,(R+rx)$ yards. By lowering the upper cage, the radii of the several coils are $R+(2n-1)\,r$, $R+(2n-3)\,r$, ... $R+[2n-(2x-1)]\,r$; the sum of which is $Rx+(2nx-x^2)\,r$. Hence, in x turns the upper cage descends $2\pi x\left\{R+(2n-1)\,r\right\}$ yards, therefore

$$d = 2\pi x\left\{R+(2n-x)\,r\right\}+2\pi x\,(R+rx) = 4\pi x\,(R+nr) = 2\pi n\,(R+nr),$$

therefore $2x = n$. Hence the two cages meet at $\pi n\,(R+\frac{1}{2}nr)$ yards from the top, or $\pi n\,(R+\frac{1}{2}nr)$ yards from the bottom.

Let $R = 1\frac{1}{2}$ yards, $r = \frac{3}{4}$ inch $= \frac{1}{16}$ yard, $d = 132$ yards. Then from (1), taking $\pi = 3\frac{1}{7}$, $n^2+72n = 1008$, therefore $n = 12$ and $x = 6$. Hence the cages will meet at $70\frac{2}{7}$ yards from the top, or $61\frac{5}{7}$ yards from the bottom.

In the above solution, n and x are both integers; but the solution of equation (1) may not always lead to an integral value of n. When this is the case, let n denote the greatest number of complete revolutions; then, α being the fractional part of the $(n+1)^{\text{th}}$ revolution, we must have

$$2\pi\left\{n\,(R+nr)+\alpha\,[R+(2n+1)\,r]\right\} = d.$$

When the upper cage has descended through the space $2\pi\alpha\,[R+(2n+1)\,r]$, the pulley contains n complete laps, and x laps $= 2\pi x\,[R+(2n-x)\,r]$. In x turns the lower cage ascends through the space $2\pi x\,(R+rx)$; and, since the upper cage has been lowered $x+\alpha+\beta$ turns, the lower cage must rise the same number of turns; and $(\alpha+\beta)$ of the $(x+1)^{\text{th}}$ lap

$$= 2\pi\,(\alpha+\beta)\,[R+(2x+1)\,r],$$

$$\therefore\ d = 2\pi x\,[R+(2n-x)\,r]+2\pi\alpha\,[R+(2n+1)\,r]+2\pi\beta\,[R+(2n-2x-1)\,r]$$
$$+2\pi x\,(R+rx)+2\pi\,(\alpha+\beta)\,[R+(2x+1)\,r]$$
$$= 2\pi n\,(R+nr)+2\pi\alpha\,[R+(2n+1)\,r],$$

$$\text{therefore}\ \frac{d}{2\pi} = 2x\,(R+nr)+2\alpha\,[R+(n+x+1)\,r]+2\beta\,(R+nr)$$
$$= n\,(R+nr)+\alpha\,[R+(2n+1)\,r].$$

Take $d = 100$ yards, $R = \frac{1}{2}$ yard, $r = \frac{1}{72}$ yard; then, putting $\alpha = 0$, we have $n^2+24n = 1145\frac{5}{11}$, therefore $n = 23$. Substituting this value of n, we find $\alpha = \frac{709}{781}$. Thus the number of turns required to draw up the lower cage $= 23\frac{709}{781}$.

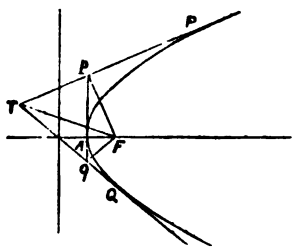
Let $\alpha = 0$, and $\beta = 0$; then $2x = n = 23$; therefore $x = 11$. Substituting these values of n, x, α, we find $\beta = \frac{1692}{38707}$; and the cages will pass each other in $11\frac{1692}{38707}$ turns; which makes the distance from the top of the shaft $62\frac{2}{3}\frac{3}{4}\frac{5}{7}$ yards.

8228. (By Capt. P. A. MacMahon, R.A.)—If A, B, a, b, c be positive integers, and $100a+10b+c$ be divisible by $10A+B$, prove that $cA^2-bAB+aB^2$ is also so divisible.

Solution by R. J. Brevor, B.A.; R. Knowles, B.A.; *and others.*

$$cA^2-bAB+aB^2 = (10A+B)\,[Ba-(10a+b)\,A]+(100a+10b+c)\,A^2.$$

8261. (By R. KNOWLES, B.A.)—From a point T tangents are drawn to a parabola whose focus is F; the tangent at the vertex meets these in p, q: prove that p, q lie on the circle described on TF as diameter.



Solutions by KATE GALE; Rev. T. GALLIERS, M.A.; *and others.*

By an elementary proposition in geometrical conics, the angles TpF and TqF are right angles. Hence a circle on TF as diameter passes through p and q.

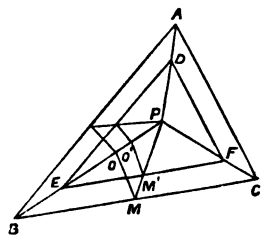
8169. (By R. TUCKER, M.A.)—P is any point in the triangle ABC; PA, PB, PC are cut in D, E, F, so that PD = k . PA, PE = k . PB, PF = k . PC; prove that, if O, O′ be the circum-centres of ABC, DEF, P, O, O′ are collinear, and P, O′ describe similar curves.

Solution by EMILY PERRIN; Prof. MATHEWS, M.A.; *and others.*

Since PD : PA = PE : PB, therefore DE is parallel to AB; similarly, all the sides of DEF are parallel and proportional to those of ABC, and O′E the radius of the circle round DEF = k . OB. Let M, M′ be the mid-points of BC : EF, then

OM : OM′ = OD : O′E = 1 : k
= BM : EM′ = MP : M′P;

therefore OO′P is a straight line, and OP : OO′ = 1 : 1 − k; hence O′, P trace out similar curves.



7704. (By ÂSÛTOSH MUKHOPÂDHYÂY, M.A., F.R.A.S.)—Prove that the loci of the middle points of series of parallel chords of the Witch of Agnesi, $xy^2 = 4a^2(2a-x)$, are, *in general*, curves of the third order; (2) the principal axis is an exception; and (3) explain how the locus degenerates into a hyperbola, and gradually into the asymptotes.

Solution by B. HANUMANTA RAU, M.A.; E. RUTTER; *and others.*

1. Let the chords be inclined at an angle α to the x axis, and let (x_1, y_1) and (x_2, y_2) be the extremities of one of the chords, (x, y) the coordinates of its middle point; then

$$y_1^2 = 4a^2\left(\frac{2a}{x_1} - 1\right), \quad y_2^2 = 4a^2\left(\frac{2a}{x_2} - 1\right) \text{ ..... } (1, 2),$$

$$y_1-y_2=(x_1-x_2)\tan\alpha,\quad y_1+y_2=2y,\text{ and }x_1+x_2=2x\ldots.\ (3,\ 4),$$

$$2y\tan\alpha=\frac{y_1^2-y_2^2}{x_1-x_2}=-\frac{8a^3}{x_1x_2}\text{ from (1) and (2),}$$

therefore $$\frac{4a^2}{x_1x_2}=-\frac{y\tan\alpha}{a},\text{ and }2a\left(\frac{1}{x_1}+\frac{1}{x_2}\right)=-\frac{xy\tan\alpha}{a^2}.$$

Again, $$2y=y_1+y_2=2a\left(\frac{2a}{x_1}-1\right)^{\frac{1}{2}}+2a\left(\frac{2a}{x_2}-1\right);$$

therefore $$\frac{y^2}{a^2}=2a\left(\frac{1}{x_1}+\frac{1}{x_2}\right)-2+2\left\{\left(\frac{2a}{x_1}-1\right)\left(\frac{2a}{x_2}-1\right)\right\}^{\frac{1}{2}}$$

$$=-\frac{xy\tan\alpha}{a^2}-2+2\left(1+\frac{xy\tan\alpha}{a^2}-\frac{y\tan\alpha}{a}\right)^{\frac{1}{2}};$$

or $$(y^2+xy\tan\alpha+2a^2)^2=4a^2(a^2+xy\tan\alpha-ay\tan\alpha),$$

or $$y^4+2xy^3\tan\alpha+y^2(x^2\tan^2\alpha+4a^2)+4a^2y\tan\alpha=0,$$

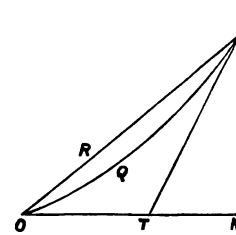
or $$y^3+2x\tan\alpha\,.\,y^2+y(x^2\tan^2\alpha+4a^2)+4a^3\tan\alpha=0,$$

a curve of the third order.

2. If $\tan\alpha=0$, the curve reduces to $y(y^2+4a^2)=0$, or $y=0$, or the axis of x, an equation of the first degree.

3704. (By Rev. A. F. Torry, M.A.)—If a curve OP always divide the triangle OPM in a constant ratio, prove that it will divide the triangle OPT in another constant ratio, PM being the ordinate to OM, and the tangent PT cutting OM in T.

Solution by the Rev. T. C. Simmons, M.A.



First, suppose PM to be at right angles to OM. Let $y=\phi(x)$ be the equation to the curve, and P any point thereon whose abscissa is h. Then, by hypothesis, $\int_0^h\phi(x)\,dx=\mu h\,\phi(h)$, where μ is some constant; hence, differentiating with respect to h, $\phi(h)=\mu\phi(h)+\mu h\,\phi'(h)$, or

$$\frac{\phi'(h)}{\phi(h)}=\frac{1-\mu}{\mu}\cdot\frac{1}{h},$$

the integration of which gives

$$\log\phi(h)=\frac{1-\mu}{\mu}\log h+\text{constant},\text{ or }\phi(h)=ch^{(1-\mu)/\mu},$$

showing that the equation to the curve is of the form $y=cx^{(1-\mu)/\mu}$.

Hence, taking x, y as coordinates of P,

$$\frac{\Delta\text{PTM}}{\Delta\text{OPM}}=\frac{\text{TM}}{\text{OM}}=\frac{y}{x}\frac{dx}{dy}=\frac{\mu}{1-\mu},$$

so that the triangle PTM, and therefore also the triangle OPT, is in a constant ratio to the triangle OPM. It follows, then, that the curvilinear area OQPR is to the triangle OPT in a constant ratio.

If PM be drawn at any other constant inclination to OM, project the figure by parallel lines so that PMO becomes a right angle; then, if there is no possibility of M lying between O and T, the above proof will apply, after which the figure may be projected back again. Thus the theorem is seen to hold also in the case when PMO is any constant *acute* angle. The curve OQP will here form what may be called an *oblique* parabolic curve.

It may be observed that $\mu : 1-\mu$ represents the ratio of the areas ORPQ, OMPQ; whence it follows that

area ORPQ : area OMPQ :: ΔPTM : ΔOPM.

7881. (By Sarah Marks.)—If mn balls have been distributed into m bags, n into each, find (1) the chance that two specified balls will be found in the same bag, and (2) what the chance becomes when r bags have been examined and found not to contain either ball.

Solution by the Rev. T. C. Simmons, M.A.; J'Oregan; *and others.*

Calling the balls A and B, the *à priori* chance that B is with A is $\frac{n}{mn-1}\frac{1}{}$, for B has $mn-1$ possible positions, $n-1$ of which are favourable. Hence *à priori* chance that B is apart from A is $\frac{n(m-1)}{mn-1}$. Now, consider the $m-r$ bags which have not been examined. If A and B are together, the chance that they occur in these bags $= \frac{m-r}{m}$. If A and B are apart, the chance that they both occur in these bags

$$= \frac{(m-2)!}{r!\,(m-r-2)!} \Big/ \frac{m!}{r!\,(m-r)!} = \frac{(m-r)(m-r-1)}{m\,(m-1)}.$$

Hence the required chance is

$$\left(\frac{n-1}{mn-1}\cdot\frac{m-r}{m}\right) \Big/ \left(\frac{n-1}{mn-1}\cdot\frac{m-r}{m} + \frac{n(m-1)}{mn-1}\cdot\frac{(m-r)(m-r-1)}{m(m-1)}\right)$$

$$= \frac{n-1}{n-1+n(m-r-1)} = \frac{n-1}{mn-nr-1}.$$

7916. (By Âsûtosh Mukhopâdhyây, M.A.)—If a comet move in an hyperbola whose semi-axis is a, eccentricity ae, show that its place at the end of t seconds, after leaving the perihelion, is determined from the equations

$$t = \frac{P}{2\pi}(a^3)^{\frac{1}{2}}[e\tan\theta - \log\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta)], \quad \tan\tfrac{1}{2}V = \left(\frac{e+1}{e-1}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\theta,$$

where V = the true anomaly reckoned from the perihelion, P = the earth's periodic time, her mean distance being unity.

Solution by Rev. T. GALLIERS, M.A.; G. G. STORR, B.A.; *and others.*

We have (see TAIT and STEELE'S *Dynamics*, 3rd ed., p. 137)

$$t = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}} \left[\log\left\{1-\left(\frac{e-1}{e+1}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\mathrm{V}\right\} \Big/ \left\{1+\left(\frac{e-1}{e+1}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}\mathrm{V}\right\} + \frac{e\,(e^2-1)^{\frac{1}{2}}\sin\mathrm{V}}{1+e\cos\mathrm{V}}\right],$$

$$\frac{\sin\mathrm{V}}{1+e\cos\mathrm{V}} = \frac{2\tan\frac{1}{2}\mathrm{V}}{1+\tan^2\frac{1}{2}\mathrm{V}+e\,(1-\tan^2\frac{1}{2}\mathrm{V})} = \frac{2\tan\frac{1}{2}\mathrm{V}}{e+1-(e-1)\tan^2\frac{1}{2}\mathrm{V}}$$

$$= \frac{1}{(e^2-1)^{\frac{1}{2}}}\tan\theta;$$

therefore $$t = \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}\left\{\log\frac{1-\tan\frac{1}{2}\theta}{1+\tan\frac{1}{2}\theta}+e\tan\theta\right\}$$

$$= \left(\frac{a^3}{\mu}\right)^{\frac{1}{2}}\left\{e\tan\theta-\log\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta\right)\right\} = \frac{\mathrm{P}}{2\pi}(a^3)^{\frac{1}{2}}\left\{e\tan\theta-\log\tan\left(\tfrac{1}{4}\pi+\tfrac{1}{2}\theta\right)\right\},$$

since $\mathrm{P} = \frac{2\pi}{\sqrt{\mu}}$ (mean distance of earth)$^{\frac{3}{2}} = \frac{2\pi}{\sqrt{\mu}}$, since this mean distance is to be taken as unity.

8248. (By Professor NASH, M.A.)—A candle, in a vertical position, is connected with a body of weight equal to its own, by a string passing over a smooth pulley, from which the weight and the candle hang freely. The candle is then lighted; prove that, if n seconds be the time for which it will burn, the weight will, in t seconds, have descended through a space

$$2ngt-\frac{gt^2}{2}+2ng\,(2n-t)\log\frac{2n-t}{2n}.$$

Solution by Professor SWAMINATHA AIYAR, B.A.; J. O'REGAN; *and others.*

The acceleration at the end of t seconds is

$$\frac{tg}{n}\Big/\left(2-\frac{t}{n}\right) = \frac{tg}{2n-t};$$

hence the velocity at the end of t seconds is

$$\int_0^t \frac{tg}{2n-t}\,dt = -gt-2ng\log\frac{2n-t}{t};$$

therefore the space passed over at the end of t seconds

$$= \int_0^t \left(-gt-2ng\log\frac{2n-t}{t}\right)dt = \text{result stated.}$$

7599. (By Professor WOLSTENHOLME, M.A., Sc.D.)—If

$$(1+2x\cos\alpha+x^2)\,/\,(1-2x\cos\alpha+x^2)\equiv z,$$

prove that

$$(1)\ \int_0^1(\log x)^6\log z\,\frac{dx}{x}=\tfrac{1}{2^8}\pi\,(\tfrac{1}{2}\pi-\alpha)(17\pi^6+34\pi^4u+18\pi^2u^2+4u^3),$$

$$(2)\ \int_0^1(\log x)^8\log z\,\frac{dx}{x}=\tfrac{1}{2^9}\pi\,(\tfrac{1}{2}\pi-\alpha)(31\pi^8+62\pi^6u+33\pi^4u^2+8\pi^2u^3+u^4),$$

where $u=\alpha(\pi-\alpha)$, and α lies between 0 and π; (3) in general,

$$\int_0^1(\log x)^{2n}\log z\,\frac{dx}{x}=4\,.\,2n!\left\{\frac{\cos\alpha}{1^{2n+2}}+\frac{\cos 3\alpha}{3^{2n+2}}+\frac{\cos 5\alpha}{5^{2n+2}}+\ldots\right\},$$

if α be any angle from 0 to π, and this can always be expressed in the form

$$\pi\,(\tfrac{1}{2}\pi-\alpha)\left(A_0\pi^{2n}+A_1\pi^{2n-2}u+\ldots+\frac{u^{2n}}{2n+1}\right).$$

Solution by B. HANUMANTA RAU, M.A.; SARAH MARKS; *and others.*

Let $2\cos\alpha=y+y^{-1}$, then $2\cos n\alpha=y^n+y^{-n}$,

$$\log z=\log\frac{(1+xy)\,(1+xy^{-1})}{(1-xy)\,(1-xy^{-1})}$$

$$=2\left\{xy+\tfrac{1}{3}(x^3y^3)+\tfrac{1}{5}(x^5y^5)+\ldots\right\}+2\left\{xy^{-1}+\tfrac{1}{3}(x^3y^{-3})+\tfrac{1}{5}(x^5y^{-5})+\ldots\right\}$$

$$=4\left\{x\cos\alpha+\tfrac{1}{3}(x^3\cos 3\alpha)+\tfrac{1}{5}(x^5\cos 5\alpha)+\ldots\right\}.$$

Substituting in the given integral (3), and remembering that

$$\int_0^1 x^m\left(\log\frac{1}{x}\right)^n=\frac{n!}{(m+1)^{n+1}},$$

we obtain $\displaystyle\int_0^1(\log x)^{2n}\log z\,\frac{dx}{x}=4\,.\,2n!\left\{\frac{\cos\alpha}{1^{2n+2}}+\frac{\cos 3\alpha}{3^{2n+2}}+\frac{\cos 5\alpha}{5^{2n+2}}+\ldots\right\}$.

8146. (By W. J. MCCLELLAND, B.A.)—If the pairs of opposite angles of any spherical quadrilateral be denoted by A and A′, B and B′, C and C′, and the diagonals by δ_1, δ_2, δ_3; prove the *algebraic* relation

$$\sin A\sin A'\cos\delta_1+\sin B\sin B'\cos\delta_2+\sin C\sin C'\cos\delta_3=0,$$

and deduce from it as particular cases the following well-known theorems:—(1) The perpendiculars of a spherical triangle are concurrent; (2) if two diagonals of a spherical quadrilateral are quadrants, the third diagonal is also a quadrant.

Solution by the PROPOSER.

Let a, b, c, d be the sides of spherical quadrilateral, d_1 and d_2 diagonals, and θ the angle between them; then we have

$$\cos a\cos c-\cos b\cos d=\sin d_1\sin d_2\cos\theta.$$

Hence $\Sigma(\cos a \cos c - \cos b \cos d) = \Sigma \sin d_1 \sin d_2 \cos\theta$,

i.e., $\Sigma \sin d_1 \sin d_2 \cos\theta \equiv 0$ .....(1),

and the supplemental theorem to (1) is the relation in question.

(β) If δ_1 and δ_2 each $= 90°$, two terms in the equation vanish; therefore $\sin C \sin C' \cos\delta_3 = 0$, *i.e.*, $\delta_3 = 90°$.

(α) The supplemental theorem to (β) and directly obtainable from (1). Therefore, &c. [See McCLELLAND's *Spher. Trig.*, Part I., pp. 148, 149.]

7345. (By J. W. L. GLAISHER, F.R.S.)—Show that (1) Mr. RAWSON's expression in Question 7416 (Vol. XLI., pp. 50 and 51), consists of a series which proceeds to infinity both ways, and is divergent for all values of x; (2) this expression, regarded as a solution of the differential equation, is therefore without meaning until such a value is given to m or n as will destroy the second series; and (3), when this is done by putting m — an uneven integer or n = an uneven integer, we obtain as two particular integrals the series U and V, and therefore $\omega AU + BV$ is the general integral of the differential equation.

Note by ROBERT RAWSON.

Mr. GLAISHER's opinion on any question respecting the solutions of Differential Equations is entitled to great respect and consideration, but, I cannot see the force of his remark that the general solution here proposed is "without meaning until such a value is given to (m) or (n) as will destroy the second series." For instance, let $m = 3$ and 5 respectively, or any odd number, then the second series becomes finite, and the particular solutions are

$$\omega_1 = \left\{x^{p+3} + \frac{a^2 x^{p+5}}{(2p+5)\,.\,2^2} + \frac{a^4 x^{p+7}}{(2p+7)(2p+5)\,.\,2^2\,.\,3^2} + \&c.\right\} + \left\{\frac{2\,(2p+3)}{a^3}\, x^{p+1}\right\},$$

$$\omega_2 = \left\{x^{p+5} + \frac{a^2 x^{p+7}}{(2p+7)\,.\,6} + \frac{a^4 x^{p+9}}{(2p+9)\,(2p+7)\,.\,6\,.\,8} + \&c.\right\}$$

$$+ \frac{2\,(2p+5)}{a^3}\, x^{p+} \left\{1 + \frac{2\,(2p+3)}{a^2 x^2}\right\};$$

the general integral is, therefore, $A\omega_1 + B\omega_2$.

It is not necessary here to enter into the knotty question of convergency and divergency of series, but, simply to remark that an infinite variety of different forms may be given to the general solution of a differential equation by giving particular values to the arbitrary constants which it contains. The series given by Mr. GLAISHER is quite correct. The general solution, therefore, which is given in the question, and from which Mr. GLAISHER's solution is easily derived may, in certain cases, be found useful.

Professor TANNER has kindly called my attention again to this question.

The series given by me for the general solution of

$$\left\{\omega_2 = a^2\omega + \frac{p(p+1)}{x^2}\omega\right\} \text{ is}$$

$$\left.\begin{aligned}\omega = A_0 x^{n-p}\Bigg(1 + \frac{a^2x^2}{n+2\,.\,m+1} + \frac{a^4x^4}{n+4\,.\,n+2\,.\,m+3\,.\,m+1} \\ + \frac{a^6x^6}{n+6\,.\,n+4\,.\,n+2\,.\,m+5\,.\,m+3\,.\,m+1} + \&c.\Bigg) \\ + A_0\frac{n(m+1)}{a^2}x^{n-p-2}\Bigg(1 + \frac{n-2\,.\,m-3}{a^2x^2} + \frac{n-4\,.\,n-2\,.\,m-5\,.\,m-3}{ax^4} \\ + \frac{n-6\,.\,n-4\,.\,n-2\,.\,m-7\,.\,m-5\,.\,m-3}{a^6x^6} + \&c.\Bigg)\end{aligned}\right\} \ldots(1);$$

where $$m = n - 2p \quad \ldots\ldots(2).$$

The above result is derived from the assumption

$$\omega = \phi(x) = \Sigma A_r x^{2r+n-p} \quad \ldots\ldots(3),$$

where (r) includes zero with all positive and negative integers. From (1) particular integrals can be had by giving particular values to the constants A_0, n, m subject to (2); and Professor Tanner, in a letter to me, has shown that, for $m = 1, 3, 5, 7$, &c., we have

$$\omega_1 = A_0 V, \quad \omega_3 = A_0\frac{2\,.\,(3+2p)}{a^2}\,.\,V, \quad \omega_5 = A_0\frac{4\,.\,2\,.\,5+2p\,.\,3+2p}{a^4}\,.\,V,$$

$$\omega_7 = A_0\frac{6\,.\,4\,.\,2\,.\,7+2p\,.\,5+2p\,.\,3+2p}{a^6}\,.\,V;\ \&c., \&c.,$$

where (V) is Mr. Glaisher's particular integral. And it is curious that for $n = 0, 2, 4, 6$, &c., we have

$$\omega_0 = A_0 U, \quad \omega_2 = A_0\frac{(1-2p)}{a^2}\,.\,U, \quad \omega_4 = A_0\frac{3-2p\,.\,1-2p}{a}\,.\,U,$$

$$\omega_6 = A_0\frac{5-2p\,.\,3-2p\,.\,1-2p}{a^6}\,.\,U, \&c.,$$

where (U) is Mr. Glaisher's other particular integral. Agreeably with the above notation, as used by Professor Tanner, it appears that

$$\omega = A\omega_{2n} + B\omega_{2n+1} \quad \ldots\ldots(4)$$

is the general integral, and the integral given in my note is a particular integral, as shown by Professor Tanner.

Mr. Glaisher's solution, as given in the paper quoted in the question, is perfectly correct, but my aim was to show the advantage of taking (r) (in $A_r x^{2r+n-p}$) negatively as well as positively.

7912. (By D. Edwardes.)—Prove that

$$\left(x^2\frac{d}{dx}\right)^n \frac{1}{x^{n-1}}\left(x\frac{d}{dx} - n\right)^r y = x^{n+1}\left(x\frac{d}{dx}\right)^r \frac{d^n y}{dx^n}.$$

Solution by the PROPOSER.

If $x = e^t$, then it can be shown that

$$\left(x^2\frac{d}{dx}\right)\frac{u}{x^{n-1}} = x\frac{d}{dt}\left(\frac{d}{dt}-1\right)\dots\left(\frac{d}{dt}-n+1\right)u,$$

and, since $$x^n\frac{d^nu}{dx^n} = \left(\frac{d}{dt}-n+1\right)\left(\frac{d}{dt}-n+2\right)\dots\left(\frac{d}{dt}-1\right)\frac{du}{dt} \quad\dots\dots(A),$$

we have $$\left(x^2\frac{d}{dx}\right)^n\frac{u}{x^{n-1}} = x^{n+1}\frac{d^nu}{dx^n}.$$

Again, putting $x = e^t$, and using the theorem (A), and the theorem

$$e^{-nt}\left(\frac{d}{dt}-n\right)^r y = \left(\frac{d}{dt}\right)^r . e^{-nt}y,$$

we have $$\left(\frac{d}{dx}\right)^n\left(x\frac{d}{dx}-n\right)^r y = \left(x\frac{d}{dx}\right)^r\frac{d^ny}{dx^n}.$$

Hence $$\left(x^2\frac{d}{dx}\right)^n\frac{1}{x^{n-1}}\left(x\frac{d}{dx}-n\right)^r y = x^{n+1}\left(\frac{d}{dx}\right)^n\left(x\frac{d}{dx}-n\right)^r y$$

$$= x^{n+1}\left(x\frac{d}{dx}\right)^r\frac{d^ny}{dx^n}.$$

7834. (By W. J. C. SHARP, M.A.)—If lines be drawn from the vertices of any tetrahedron ABCD through the same point P, to meet the opposite faces in A′, B′, C′, D′ respectively: show that

$$\frac{PA'}{AA'}+\frac{PB'}{BB'}+\frac{PC'}{CC'}+\frac{PD'}{DD'} = 1.$$

Solution by HANUMANTA RAU, M.A.; E. RUTTER; *and others.*

If a and h be the distances of P and A from the face BCD, we have

$$PA' : AA' = a : h = \text{vol. PBCD} : \text{vol. ABCD}.$$

Similarly PB′/BB′ = vol. PACD / vol. BACD, and so on.

But vol. PBCD + vol. PACD + vol. PABD + vol. PABC = vol. ABCD; hence the stated result follows.

8072. (By ASUTOSH MUKHOPÂDHYÂY, M.A., F.R.A.S.)—Prove that

$$\int_b^a \phi(x)\,dx = \frac{a-b}{\alpha-\beta}\int_\beta^\alpha \phi\left(\frac{a-b}{\alpha-\beta}x+\frac{ab-\beta a}{\alpha-\beta}\right)dx.$$

Solution by D. Edwardes; Belle Easton; *and others.*

Let $x = Ay + B$, then $a = A\alpha + B$, $b = A\beta + B$, whence

$$A = \frac{a-b}{\alpha-\beta}, \quad B = \frac{ab-\alpha\beta}{\alpha-\beta}, \quad \phi(x) = \phi\left(\frac{a-b}{\alpha-\beta}y + \frac{ab-\alpha\beta}{\alpha-\beta}\right),$$

$$\int_b^a \phi x\, dx = \frac{a-b}{\alpha-\beta}\int_\beta^\alpha \phi\left(\frac{a-b}{\alpha-\beta}y + \frac{ab-\alpha\beta}{\alpha-\beta}\right) dy.$$

7979. (By R. Knowles, B.A.)—If Δ be the area of the triangle ABC, and Δ_1 that formed by joining the ex-centres, show that

$$\Delta_1 : \Delta = 8sR^2 : abc.$$

Solution by G. G. Storr, B.A.; E. Rutter; *and others.*

If $s_1 = s-a$, &c., we have $\Delta_1 = \Delta + \frac{1}{2}ar_1 + \frac{1}{2}br_2 + \frac{1}{2}cr_3$;

therefore $$\frac{\Delta_1}{\Delta} = 1 + \frac{a}{2s_1} + \frac{b}{2s_2} + \frac{c}{2s_3} = \frac{abc}{2s_1s_2s_3} = \frac{abcs}{2\Delta^2} = \frac{8sR^2}{abc};$$

since $$R = \frac{abc}{4\Delta} \text{ and } \frac{8R^2}{abc} = \frac{abc}{2\Delta^2}.$$

8077. (By Asûtosh Mukhopâdhyây, M.A., F.R.A.S.)—Upon two lines AB, AB′, two points C, C′ are taken, such that AC . AC′ = BC . B′C′. Find the curve which is constantly touched by a straight line passing through C, C′.

Solution by Asparagus; Rev. T. Galliers, M.A.; *and others.*

Take AB, AB′ as axes of coordinates; and let AB = a, AB′ = b, AC = p, AC′ = q; then the equation of CC′ is $\frac{x}{p} + \frac{y}{q} = 1$, when p, q are subject to the condition $pq = (a-p)(b-q)$ or $1 = \frac{p}{a} + \frac{q}{b}$. The equation to CC′ may be written

$$\left(\frac{x}{p} + \frac{y}{q}\right)\left(\frac{p}{a} + \frac{q}{b}\right) = 1, \text{ or } \frac{x}{a} + \frac{y}{b} - 1 + \frac{p}{q}\cdot\frac{y}{a} + \frac{q}{p}\cdot\frac{x}{b} = 0,$$

involving the parameter p/q only. The envelope is then the parabola $\left(\frac{x}{a} + \frac{y}{b} - 1\right)^2 = \frac{4xy}{ab}$, *i. e.* the parabola touching AB, AB′ in the points B, B′.

7528. (By W. J. C. SHARP, M.A.)—Deduce HERMITE's law of reciprocity from the identity

$$\frac{(1-x^{n+1})(1-x^{n+2})\dots(1-x^{n+\theta})}{(1-x)(1-x^2)\dots(1-x^{\theta})} \equiv \frac{(1-x^{\theta+1})(1-x^{\theta+2})\dots(1-x^{n+\theta})}{(1-x)(1-x^2)\dots(1-x^{n})}.$$

Solution by the PROPOSER.

The weight of a covariant of the binary n-ic of order p in x and y and θ in the coefficients is $\frac{1}{2}(n\theta+p)$, which is also that of a covariant of the θ-ic of the same order in x and y and of the n^{th} order in the coefficients; and the number of covariants of the first class is the difference between the coefficients of $x^{\frac{1}{2}(n\theta+p)}$ and $x^{\frac{1}{2}(n\theta+p)-1}$ in the first expression (SALMON's *Higher Algebra*, p. 272), and the number of covariants of the second class is the difference of the coefficients of the same powers of x in the second and equivalent expression, which proves HERMITE's theorem in the more general form (SALMON's *Higher Algebra*, Art. 154), "that of covariants of any given degree in x and y, the n-ic possesses as many of the θ^{th} order in the coefficients as the θ-ic has of the n^{th} order in the coefficients."

8084. (By Professor SÁRADÁRANJAN RÂY, M.A.)—If P_n be a zonal harmonic of the n^{th} order, prove that

$$P_n = C(1-\mu^2)^{\frac{1}{2}n} + n(1-\mu^2)^{\frac{1}{2}n}\int_0^{\mu} P_{n-1}(1-\mu^2)^{-\frac{1}{2}(n+2)}\,d\mu,$$

where $C = 0$ if n be odd, and $= (-1)^{\frac{1}{2}n}\, n!\, 2^{-n}[\frac{1}{2}n!]^{-2}$ if n be even.

Solution by D. EDWARDES; EMILY PERRIN; *and others.*

We have $(1-\mu^2)\frac{dP_n}{d\mu} + n\mu P_n = n P_{n-1}$ (TODHUNTER's *Functions*, Art. 58),

therefore $$\frac{1}{(1-\mu^2)^{\frac{1}{2}n}}\frac{dP_n}{d\mu} + \frac{n\mu P_n}{(1-\mu^2)^{\frac{1}{2}n+1}} = \frac{nP_{n+1}}{(1-\mu^2)^{\frac{1}{2}n+1}}.$$

Integrating between 0 and μ,

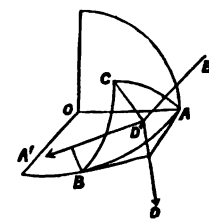
$$\frac{P_n}{(1-\mu^2)^{\frac{1}{2}n}} = \left[\frac{P_n}{(1-\mu^2)^{\frac{1}{2}n}}\right]^{\mu=0} + n\int_0^{\mu}\frac{P_{n-1}}{(1-\mu^2)^{\frac{1}{2}n+1}}\,d\mu,$$

and the lowest term in P_n is $(-1)^{\frac{1}{2}n}\dfrac{1\cdot3\cdot5\dots n-1}{2\cdot4\dots n}$, if n be even, and is $(-1)^{\frac{1}{2}(n-1)}\dfrac{3\cdot5\dots n}{2\cdot4\dots(n-1)}\,\mu$, if n be odd, and therefore, &c.

8154. (By W. J. C. SHARP, M.A.)—If planes be drawn touching the sphere, circumscribed about a given tetrahedron, at its vertices, these will contain another tetrahedron: determine its angles, plane, and dihedral.

Solution by A. Gordon, M.A.; G. G. Storr, B.A.; *and others.*

Let the angles subtended at the centre by the sides BC, CA, AB, AD, &c., be α, β, γ, α', β', γ'. $\left\{ \alpha = 2\sin^{-1}\frac{a}{2R}, \text{ &c., where } a, b, c, \text{ &c. are the lengths BC, CA, &c....,} \text{ and where } R^2 = \frac{s\,.\,s-ad\,.\,s-be\,.\,s-c}{36V^2} \right\}$.



Let the tangent planes at ABC, cut in D', &c., so that A'B'C'D' is the new tetrahedron.

1°. The dihedral angles will be $\pi-\alpha, \pi-\beta$, &c., for the angle between the tangent plane at B and tangent plane at C, which intersect along D'A', $= \pi - \widehat{COB}$, &c., &c.

2°. The plane angles at D', for example, will be $\pi-\widehat{C}, \pi-\widehat{B}, \pi-\widehat{A}$, in which $\pi-C$ will correspond to the angle between the intersections of tangent planes at A and B with that at C, &c. $= A'D'B' = \pi - C$;

$$\left[\cos C = \frac{\cos\gamma - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}, \text{ &c.}\right]$$

7978. (By Sarah Marks.)—If O be the in-centre, O' the circum-centre and P the ortho-centre, R the circum-radius of the triangle ABC, prove that the area (Δ_1) of the triangle OO'P is

$$\Delta_1 = -2R^2 \sin\tfrac{1}{2}(B-C)\sin\tfrac{1}{2}(C-A)\sin\tfrac{1}{2}(A-B).$$

Solution by Rev. T. Galliers, M.A., *and* B. Hanumanta Rau, M.A.

The trilinear coordinates of O, O', P are (r, r, r), $(R\cos A, R\cos B, R\cos C)$, $(2R\cos B\cos C, 2R\cos C\cos A, 2R\cos A\cos B)$;

therefore $\Delta_1 = \frac{R^3 r}{\Delta}\begin{vmatrix} 1, & 1, & 1 \\ \cos A, & \cos B, & \cos C \\ \cos B\cos C, & \cos C\cos A, & \cos A\cos B \end{vmatrix}$,

$$= \frac{R^3 r}{\Delta}[\cos A(\cos^2 B - \cos^2 C) + \cos B(\cos^2 C - \cos^2 A) + \cos C(\ )]$$

$$= \frac{R^3 r}{\Delta}(\cos A - \cos B)(\cos B - \cos C)(\cos C - \cos A)$$

$$= \frac{8R^3 r}{\Delta}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\,.\,\sin\frac{B-A}{2}\sin\frac{C-B}{2}\sin\frac{A-C}{2};$$

and $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \frac{S\,.\,\Delta}{abc} = \frac{\Delta^2}{abc\,.\,r} = \frac{\Delta}{4Rr}$;

therefore $\Delta_1 = -2R^2\sin\frac{B-C}{2}\sin\frac{C-A}{2}\sin\frac{A-B}{2}$.

[For a geometrical solution, see Vol. xliii., p. 60.]

8034. (By Âsûtosh Mukhopâdhyây, M.A., F.R.A.S.)—If

$$x = [y\,(y-\lambda)]^{\frac{1}{2}} \pm \lambda \log \frac{[\sqrt{y}-\sqrt{\lambda}]^{\frac{1}{2}} + [\sqrt{y}+\sqrt{\lambda}]^{\frac{1}{2}}}{[\sqrt{y}-\sqrt{\lambda}]^{\frac{1}{2}} - [\sqrt{y}+\sqrt{\lambda}]^{\frac{1}{2}}},$$

prove that $\left(\frac{d^2r}{dy^2}\right)^2 = \frac{\lambda^2}{4} \cdot \frac{1}{y\,(y-\lambda)^3}$ or $\frac{\lambda^2}{4} \cdot \frac{1}{y^3\,(y-\lambda)}$.

Solution by G. G. Storr, B.A.; D. Edwardes; *and others.*

We have $x = (y^2-\lambda y)^{\frac{1}{2}} \pm \lambda \left\{\log [y^{\frac{1}{2}} + (y-\lambda)^{\frac{1}{2}}] - \log (-\lambda^{\frac{1}{2}})\right\}$;

therefore $\frac{dx}{dy} = \left(1-\frac{\lambda}{y}\right)^{-\frac{1}{2}}$ or $\left(1-\frac{\lambda}{y}\right)^{\frac{1}{2}}$,

therefore $\frac{d^2x}{dy^2} = -\frac{\lambda}{2y^2}\left(1-\frac{\lambda}{y}\right)^{-\frac{1}{2}}$ or $\frac{\lambda}{2y^2}\left(1-\frac{\lambda}{y}\right)^{-\frac{1}{2}}$.

[If we put $y+h$ for y, and expand

$$(y+h)^{\frac{1}{2}}\,(y-\lambda+h)^{\frac{1}{2}} \pm \lambda \log \left\{(y+h)^{\frac{1}{2}} + (y-\lambda+h)^{\frac{1}{2}}\right\}$$

in powers of h, we find, for the coefficient of h^2,

$$-\frac{(y-\lambda)^{\frac{1}{2}}}{8y^{\frac{3}{2}}} - \frac{y^{\frac{1}{2}}}{8\,(y-\lambda)^{\frac{3}{2}}} + \frac{1}{4} \cdot \frac{1}{y^{\frac{1}{2}}\,(y-\lambda)^{\frac{1}{2}}} - \frac{\lambda}{8} \cdot \frac{2y-\lambda-y^{\frac{1}{2}}\,(y-\lambda)^{\frac{1}{2}}}{y^{\frac{3}{2}}\,(y-\lambda} - \frac{\lambda}{8y\,(y-\lambda)},$$

or

$$\frac{-\lambda}{4y^{\frac{1}{2}}(y-\lambda)^{\frac{3}{2}}},$$

and this, by Taylor's theorem, is $2\,d^2x/dy^2$. Similarly, taking the lower sign, we find the coefficient of $\frac{1}{2}h^2$ to be $\frac{\lambda}{2} \cdot \frac{1}{y^{\frac{3}{2}}\,(y-\lambda)^{\frac{1}{2}}}$.]

8032. (By Satis Chandra Ray.)—If S be the area of any triangle, and S_n that of its n^{th} pedal, prove that

$$\frac{S_n}{S} = (-2)^{2n} \sin (2^n-1)\,A \,.\, \sin (2^n-1)\,B \,.\, \sin (2^n-1)\,C.$$

Solution by B. Hanumanta Rau, M.A.

If A, B, C be the angles of the original triangle, the angles of the pedal triangles will be $\pi-2A$, $\pi-2B$, $\pi-2C$; $4A-\pi$, $4B-\pi$, $4C-\pi$; $3\pi-8A$, $3\pi-8B$, $3\pi-8C$; $16A-5\pi$, $16B-5\pi$, $16C-5\pi$. It is also known that

$$\frac{S_1}{S} = 2\cos A \cos B \cos C, \quad \frac{S_2}{S_1} = 2\,(-1)^3 \cos 2A \cos 2B \cos 2C,$$

$$\frac{S_3}{S_2} = 2\,(-1)^3 \cos 4A \cos 4B \cos 4C, \ldots\ldots$$

$$\frac{S_n}{S_{n-1}} = 2\,(-1)^3 \cos 2^{n-1}A \cos 2^{n-1}B \cos 2^{n-1}C.$$

But $\sin 2^n A = 2^n \sin A \cos A \cos 2A \ldots\ldots \cos 2^{n-1}A$.

Therefore $\dfrac{S_n}{S} = 2^n(-1)^{3n}\dfrac{\sin 2^nA}{2^n \sin A}\cdot\dfrac{\sin 2^nB}{2^n \sin B}\cdot\dfrac{\sin 2^nC}{2^n \sin C}$,

which gives the stated result.

7731. (By J. McDowell, M.A.)—On the three sides of a triangle ABC, segments of circles are described, internally, capable of containing angles respectively equal to the supplements of the opposite angles, and intersecting in P. Find the lengths of the straight lines PA, PB, PC in terms of the sides and angles of the given triangle.

Solution by (1) B. Hanumanta Rau, M.A.; (2) *the* Proposer.

1. Let x, y, z be the required lengths, then, by Wolstenholme's *Problems*, No. 582,

$$\frac{x \sin A}{\sin 2A} = \frac{y \sin B}{\sin 2B} = \frac{z \sin C}{\sin 2C},$$

or $$\frac{x}{\cos A} = \frac{y}{\cos B} = \frac{z}{\cos C};$$

and $$y^2 + z^2 - 2yz\cos(\pi - A) = a^2,$$

therefore

$$\frac{x^2}{\cos^2 A}\{\cos^2 B + \cos^2 C + 2\cos A \cos B \cos C\} = a^2,$$

or $$x = \frac{a\cos A}{\sin A} = a\cot A, \quad y = b\cot B, \quad \text{and} \quad z = c\cot C.$$

This result is obvious from the fact that P is, by construction, the orthocentre of the triangle ABC.

[If we make $\angle BCD = PCA$ and $\angle CPD = BAC$, the triangle PCD will be similar to ABC, and triangle BCD to ACP; hence we have

$$\frac{PD}{\sin C} = \frac{z}{\sin B}, \quad \frac{BD}{x} = \frac{BC}{AC} = \frac{\sin A}{\sin B}, \quad \frac{PD}{\sin PBD} = \frac{DB}{\sin BPD} = \frac{BP}{\sin BDP};$$

therefore $$\frac{z \sin C}{\sin B \sin(BPA - C)} = \frac{x \sin A}{\sin B \sin(BPC - A)} = \frac{y}{\sin(APC - B)},$$

or $$\frac{z \sin C}{\sin 2C} = \frac{x \sin A}{\sin 2A} = \frac{y \sin B}{\sin 2B},$$

and we may proceed as in the foregoing solution.]

2. It is easy to see that P is the orthocentre of the triangle ABC, and therefore, if $PA = x$, $PB = y$, $PC = z$, and O be centre of circumscribed circle, and OD perpendicular to BC,

$$x = 2OD = 2R\cos A = \frac{a(b^2 + c^2 - a)^2}{4S}.$$

Similarly $y = 2R\cos B$, $z = 2R\cos C$. Hence the above values of x, y, z, in terms of a, b, c, will furnish the solution of the equations

$$bc(y^2+z^2)+(b^2+c^2-a^2)yz = a^2bc, \quad ca(z^2+x^2)+(c^2+a^2-b^2)zx = b^2ca,$$
$$ab(x^2+y^2)+(a^2+b^2-c^2)xy = c^2ab,$$

for *any* values of a, b, c.

7156. (By A. McMurchy, B.A.)—If $x = e^\theta\cos\theta$, $y = e^\theta\sin\theta$, prove that $\dfrac{d^m}{dx^m}\cdot\dfrac{d^n}{dy^n}\cdot\rho = -(m+n-1)!\,e^{-(m+n)\rho}\cos[(m+n)\theta+\frac{1}{2}\pi(2m+n)]$,

or $\qquad (m+n-1)\,e^{-(m+n)\rho}\sin[(m+n)\theta+\frac{1}{2}\pi(2m+n-1)]$.

Solution by the Proposer.

Square, take logs and differentiate, then

$$\frac{d\rho}{dx} = \frac{x}{x^2+y^2}, \quad \frac{d^m\rho}{dx^m} = \frac{(m-1)!\,(-1)^{m-1}}{2}\left\{\frac{1}{(x+y\sqrt{-1})^m}+\frac{1}{(x-y\sqrt{-1})^m}\right\},$$

$$\frac{d^m}{dx^m}\cdot\frac{d^n}{dy^n}\cdot\rho = \frac{(m+n-1)!\,(-1)^{m-1}}{2}\left\{\frac{(-1)^{n+\frac{1}{2}n}}{(x+y\sqrt{-1})^{m+n}}+\frac{(-1)^{\frac{1}{2}n}}{(x-y\sqrt{-1})^{m+n}}\right\},$$

(n even) $\quad = (-1)^{m+\frac{1}{2}n-1}(m+n-1)!\,e^{-(m+n)\rho}\cos(m+n)\theta,$

(n odd) $\quad = (-1)^{m+\frac{1}{2}n-1}(m+n-1)!\,e^{-(m+n)\rho}\sin(m+n)\theta,$ = &c.

7906. (By J. Brill, B.A.)—The tangents to a parabola at the points P, Q, R meet the tangent at the vertex in the points X, Y, Z, respectively; prove that

$$SP\,.\,YZ\,.\,\sin ASP + SQ\,.\,ZX\,.\,\sin ASQ + SR\,.\,XY\,.\,\sin ASR = 0 \ldots (1),$$
$$AS\,(SP\,.\,YZ\,.\,\cos ASP + SQ\,.\,ZX\,.\,\cos ASQ + SR\,.\,XY\,.\,\cos ASR)$$
$$= YZ\,.\,ZX\,.\,XY \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

subject to the usual convention concerning the signs of lengths measured along the same straight line.

Solution by R. Knowles, B.A.; Rev. T. Galliers, M.A.; *and others.*

Let the equation to the parabola be $y^2 = 4ax$, and x_1y_1, x_2y_2, x_3y_3 the coordinates of P, Q, R respectively; the values of AX, AY, AZ, YZ, ZX, and XY are severally $\frac{1}{4}y_1$, $\frac{1}{4}y_2$, $\frac{1}{4}y_3$, $\frac{1}{4}(y_2-y_3)$, $\frac{1}{4}(y_3-y_1)$, $\frac{1}{4}(y_1-y_2)$, and $SP\,.\,\sin ASP = y_1$, &c.; therefore (1) becomes

$$\tfrac{1}{4}[y_1(y_2-y_3)+y_2(y_3-y_1)+y_3(y_1-y_2)] = 0.$$

Again, $SP\,.\,\cos ASP = a-x_1$, &c.; therefore

$$(2) = -\tfrac{1}{4}a[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$$
$$= -\tfrac{1}{8}[y_1^2(y_2-y_3)+y_2^2(y_3-y_1)+y_3^2(y_1-y_2)] = \tfrac{1}{8}(y_1-y_2)(y_3-y_1)(y_2-y_3)$$
$$= YZ\,.\,ZX\,.\,XY.$$

8255. (By Rev. T. C. Simmons, M.A.)—A triangle is formed by three random tangents drawn to a given circle. Prove that the odds are 15 to 1 against its being acute-angled.

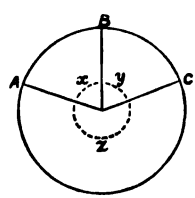
Solutions by (1) Professor Swaminatha Aiyar, B.A.; (2) *the* Proposer.

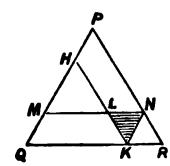
1. Let one of the tangents be fixed; let the circumference be divided into $4n$ equal parts; and let the point of contact of the fixed tangent be one of the points of division. The number of ways of choosing two points out of the $4n-1$ remaining points, such that they shall be distant from each other and from the fixed point by more than a fourth part of the circumference, is $\frac{1}{2}(n-2)(n-1)$. The ratio of this number to the number of ways of choosing two points in any manner out of the $4n-1$ is $\frac{(n-2)(n-1)}{(4n-2)(4n-1)}$; and when n is infinite this ratio is equal to $\frac{1}{16}$, which is evidently the chance, when the circle is *inscribed*, of the triangle being acute-angled. And, since to every such triangle there correspond three others, with parallel sides, having the circle escribed, it is plain that the above ratio must be multiplied by 4, thereby giving $\frac{1}{4}$ as the whole chance that three random tangents form an acute-angled triangle. (See the figure corresponding to the last solution of Quest. 8055, Vol. xliv., p. 56.)

2. Let the three points of contact be denoted by A, B, C; A being supposed fixed. Take x, y, z to represent the angles subtended at the centre by the arcs AB, BC, CA respectively, so that $x+y+z=2\pi$. It is then clear that the following four cases will give acute-angled triangles—

$$\left.\begin{matrix} x > \frac{1}{2}\pi \\ y > \frac{1}{2}\pi \\ z > \frac{1}{2}\pi \end{matrix}\right\} \ldots\ldots(\alpha), \quad \left.\begin{matrix} x < \frac{1}{2}\pi \\ y < \frac{1}{2}\pi \\ z < \frac{3}{2}\pi \end{matrix}\right\} \ldots\ldots(\beta), \quad \left.\begin{matrix} x < \frac{1}{2}\pi \\ y < \frac{3}{2}\pi \\ z < \frac{1}{2}\pi \end{matrix}\right\} \ldots\ldots(\gamma), \quad \left.\begin{matrix} x < \frac{3}{2}\pi \\ y < \frac{1}{2}\pi \\ z < \frac{1}{2}\pi \end{matrix}\right\} \ldots\ldots(\delta).$$

Taking rectangular axes Ox, Oy, Oz, let the plane $x+y+z=2\pi$ cut them respectively in P, Q, R. Then the chance of the occurrence of (α) will be represented by the ratio of the area intercepted on PQR between the planes $x=\frac{1}{2}\pi$, $y=\frac{1}{2}\pi$, $z=\frac{1}{2}\pi$, to the whole area of PQR, which is evidently $\frac{1}{16}$. Similarly, if the planes $x=\frac{1}{2}\pi$, $y=\frac{1}{2}\pi$, $z=\frac{3}{2}\pi$ cut PQR





in three lines denoted respectively in the figure by MN, HK, KN, the chance of the occurrence of (β) will be represented by the ratio of the triangles KLN, PQR, which is again clearly $\frac{1}{16}$. In the same way the chance of the occurrence of (γ) or (δ) is seen to be in each case $\frac{1}{16}$. And as in every other case the triangle will be obtuse-angled, the required chance that the three random tangents will form an acute-angled triangle is $\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{1}{4}$, or the odds are 3 to 1 against.

8188. (By Professor LONGCHAMPS.)—Résoudre et discuter l'équation

$$\sin(45° + \tfrac{3}{2}x) = h \sin(45° - \tfrac{1}{2}x).$$

Solution by W. J. GREENSTREET, B.A.; BELLE EASTON; *and others.*

$$\frac{\sin(45° + \frac{3}{2}x)}{\sin(45° - \frac{1}{2}x)} = h, \quad \frac{2\sin\frac{1}{2}(90° + x)\cos x}{2\cos\frac{1}{2}(90° + x)\sin x} = \frac{1+h}{1-h};$$

$$\tan\tfrac{1}{2}(90° + x)\cot x = \frac{1+h}{1-h}, \quad \frac{1+\sin x}{\sin x} = \frac{1+h}{1-h}, \quad \frac{1}{\sin x} = \frac{1+h}{1-h} - 1 = \frac{2h}{1-h}.$$

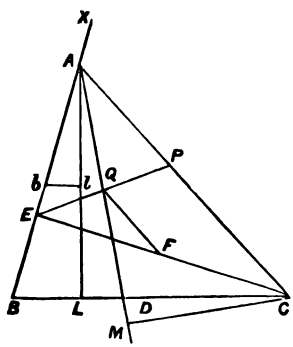
Therefore $\sin x = \dfrac{1-h}{2h}$, therefore $x = n\pi + (-1)^n \sin^{-1}\dfrac{1-h}{2h}$.

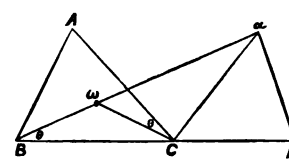
Now $\dfrac{1-h}{2h}$ is never >1, therefore $1-h$ is either $<$ or $= 2h$, therefore $1 <$ or $= 3h$, therefore h is $>$ or $= \frac{1}{3}$, similarly $-\frac{1}{3}$ is a limit.

8349. (By Professor NEUBERG.)—Lorsque les sommets B, C d'un triangle ABC sont fixes, et que le sommet A décrit une droite donnée BX; (1) le centre de gravité et le centre du cercle des neuf points décrivent des droites, et la droite qui joint ces points enveloppe une parabole; (2) la symédiane partant de A enveloppe une conique ayant un foyer en C, et le centre des symédianes décrit une conique; (3) l'un des points de Brocard décrit une circonférence, et l'autre une podaire de conique.

Solution by R. F. DAVIS, M.A.

1. Draw CE perpendicular to BX, and let D, F be the middle points of BC, CE. Let O, H, N, G be respectively the circumcentre, orthocentre, nine-points centre, and centre of gravity of the triangle ABC.





Since 3DG = DA, the locus of G is a straight line parallel to BX; since ND = NE, the locus of N is a straight line bisecting DE at right angles; the locus of O is a straight line through D perpendicular to BC, and the locus of H is the perpendicular CE. Now, since N, G lie on OH and divide it in a fixed ratio, the envelope of ONGH is a parabola touching the four straight lines which are the several loci of O, N, G, H.

2. Through E draw the antiparallel EP to BC, meeting AC in P. Then, if Q be its middle point, AQ is the symmedian line through A. Let CM be perpendicular to AM; then A, E, M, C are concyclic and $\angle EMQ = \angle ECA = \angle EFQ$, and M lies on the fixed circle through E, F containing an $\angle EQF = \angle EPC = \pi - B$. Thus the envelope of AQ is a conic whose focus is C.

If b, l be the middle points of AB and the perpendicular AL on BC, then K, the symmedian centre, is the intersection of Dl, Fb. But bl is always parallel to BC while its extremities lie on fixed straight lines through B. Hence, by MacLaurin's theorem, the locus of K is a conic passing through B, D, F.

3. Let ω, ω' be the Brocard-points. Since $\angle B\omega' C = \pi - B$ the locus of ω' is a segment of a circle passing through B, C.

Produce ω to α, so that $B\omega \cdot B\alpha = BC^2$, and draw $\alpha\beta$, making with BC produced an $\angle \alpha\beta C = B$. Then, since BC touches the circle about $C\omega\alpha$, $\angle \alpha C\beta = C$ and the triangles ABC, $\alpha\beta C$ are similar. Since

$$\sin A \cdot \sin B \cdot \sin (C - \theta) = \sin \theta \cdot \sin^2 C,$$

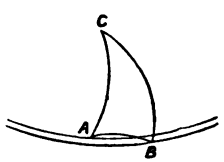
where θ is the Brocard-angle, it may be shown that $\alpha\beta^2 = BC \cdot C\beta$, and the locus of α is a parabola having $C\beta$ for a diameter. Hence, since ω, α are inverse points, the locus of ω is the inverse of a parabola (or the pedal of its polar reciprocal) with respect to B.

8126. (By Professor Hudson, M.A.)—In sailing in a great circle from a port in north latitude λ to another in north latitude $\lambda - \delta$, the difference of longitude being 2ϕ, show that part of the shorter course will lie further north than the northernmost port, unless δ exceed

$$\tan^{-1} \left[\sin 2\lambda \sin^2 \phi \,/\, (1 - 2 \sin^2 \lambda \sin^2 \phi) \right].$$

Solution by Professor Mahendra Nath Ray, M.A., B.L.

Let C be the north pole, A and B the two ports, A being the northernmost one, and AB the shorter course from A to B; then a part of AB will lie further north than A if the shortest line from C to AB, which is perpendicular to AB, fall between A and B, *i.e.*, if A be an acute angle.



Now $\cot A = 1 \,/\, \tan \left[\frac{1}{2} (A + B) + \frac{1}{2} (A - B) \right]$, and, by substituting for $\tan \frac{1}{2} (A + B)$ and $\tan \frac{1}{2} (A - B)$ the values given by Napier's analogies, this expression may be shown to be

$$\sin (a + b) - \sin (a - b) \cot^2 \tfrac{1}{2} C \,/\, 2 \sin a \cot \tfrac{1}{2} C.$$

But, by supposition, $a = 90° - \lambda + \delta$, $b = 90° - \lambda$, $C = 2\phi$,
therefore $\cot A = \sin(2\lambda - \delta) - \sin\delta \cot^2\phi \,/\, 2\cos(\lambda - \delta)\cot\phi$
$= \cos\delta\left[\sin 2\lambda - (\cos 2\lambda + \cot^2\phi)\tan\delta\right] / 2\cos(\lambda - \delta)\cot\phi$.
In order that A may be an acute angle, cot A must be positive; therefore $(\cos 2\lambda + \cot^2\phi)\tan\delta < \sin 2\lambda$, or δ must be less than
$\tan^{-1}\sin 2\lambda \,/\, (\cos 2\lambda + \cot^2\phi)$ or $\tan^{-1}\left[\sin 2\lambda \sin^2\phi \,/\, (1 - 2\sin^2\lambda\sin^2\phi)\right]$;
and if δ exceed this quantity, no part of the course will lie further north than A.

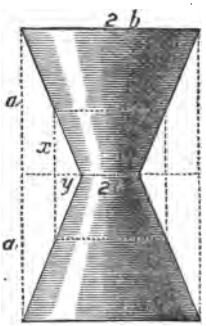
[This follows at once from $\cot a \sin b = \cot A \sin C + \cos b \cos C$, where $a = 90° - \lambda + \delta$, $b = 90° - \lambda$, $C = 2\phi$.]

8195. (By the Editor.)—If n men pay equally towards the purchase of a grindstone of radius a, and respective thicknesses $2b$, $2c$ at circumference and centre, show (1) that if the rate of decrease of thickness is uniform, and $m = (b-c)/a$, the radius (x) of the portion of the stone remaining for the n^{th} man to grind down for his share will be given by the equation $nx^2(2mx + 3c) = a^2(2b + c)$; (2) apply this to find what part of a stone, 24 inches in radius, 6 inches thick at circumference, and 4 inches at centre, must be ground down by each of 3 men paying equally; and solve the problem when the grindstone is of the form of a double concave lens, (3) spherical, or (4) paraboloidal.

Solution by Rev. T. Galliers, M.A.

1. Let the black lines in the figure indicate a section of the grindstone made by a plane passing through its axis of rotation. Then it is clear that, if V be the volume of the whole stone, V_1 the volume of a cylinder of which the height is $2b$ and radius of base is a, and V_2 the volume of a cone of which the height is $b-c$ and radius of base is a, we shall have

$$V = V_1 - 2V_2 = \pi a^2 \cdot 2b - \tfrac{2}{3}\pi a^2 (b-c) = \tfrac{2}{3}\pi a^2 (2b + c).$$



Again, V_n = volume of portion of stone remaining for n^{th} man to grind down = (volume of cylinder of height $2(c+y)$ and radius of base x) − twice (volume of cone of height y and radius of base x); but $y : x = b - c : a = m : 1$,
therefore $V_n = \pi x^2 \cdot 2(y + c) - \tfrac{2}{3}\pi x^2 y = 2\pi x^2 (mx + c) - \tfrac{2}{3}\pi m x^3$
$= \tfrac{2}{3}\pi \cdot x^2 (2mx + 3c)$.
Now, by the question $n \cdot V_n = V$;
therefore $nx^2(2mx + 3c) = a^2(2b + c)$ ..... (α).

2. Here $a = 24$, $b = 3$, $c = 2$, $n = 3$; therefore $m = \frac{1}{24}$;
Thus equation (α) becomes $x^3 + 3^7 \cdot 2^3 x^2 = 3^2 \cdot 2^{11}$,
or, putting $x = y^{-1}$, $y^3 - \dfrac{1}{2^8} \cdot y - \dfrac{1}{3 \cdot \times 2^{11}} = 0$ ..... (β).

Now the roots of the equation $y^3 - qy - r = 0$, are known to be

$$2(\tfrac{1}{3}q)^{\frac{1}{2}} \cos \tfrac{1}{3}\alpha, \quad 2(\tfrac{1}{3}q)^{\frac{1}{2}} \cos \tfrac{1}{3}(2\pi \pm \alpha), \text{ where } \cos\alpha = \tfrac{1}{2}r\left(\frac{3}{q}\right)^{\frac{3}{2}}.$$

In (β) we find $\cos\alpha = \frac{1}{2}\sqrt{3} = \cdot 5774$, whence $\alpha = 54° 44'$;

therefore $y = 2(\frac{1}{3}q)^{\frac{1}{2}} \cos \frac{1}{3}\alpha$, therefore $x = \dfrac{1}{y} = \dfrac{2^3 \cdot 3^{\frac{1}{2}}}{\cos(18° 14')}$

$= 14 \cdot 59$ inches nearly = radius of portion of stone left for the third man to grind down.

To find the radius of the portion left for the second man to grind down, we must write, in (α), $n = \frac{2}{3}$; thus, instead of (β), we shall have

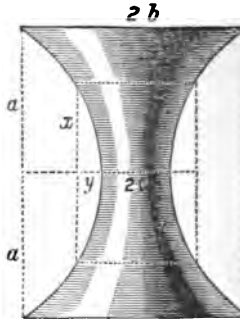
$$y^3 - \frac{1}{2^9}\, y - \frac{1}{3^3 \times 2^{12}} = 0,$$

whence $\cos\alpha = \frac{1}{3}\sqrt{6} = \cdot 8165$, $\alpha = 35° 15'$ nearly,

$$x = \frac{1}{y} = \frac{2^3\sqrt{6}}{\cos(11° 45')} = 20 \cdot 01 \text{ inches nearly,}$$

= the radius of portion left for second man to grind down.

3. Let the form of the stone be that of a double concave lens, r being the radius of the spherical surface.



Now it may be proved that the volume of a segment of a sphere whose height is h, and the radius of its base ρ, is $\frac{1}{6}\pi h(h^2 + 3\rho^2)$;
also in the figure $a^2 = 2r(b-c) - (b-c)^2$,

or $$r = \frac{a^2 + (b-c)^2}{2(b-c)} = a\left(\frac{1+m^2}{2m}\right);$$

also $$x^2 = 2ry - y^2,$$

and the smaller value of $y = r - (r^2 - x^2)^{\frac{1}{2}} \ldots(\gamma)$,

volume of whole stone

$$= \pi a^2 - 2b - \tfrac{1}{3}\pi(b-c)\left\{(b-c)^2 + 3a^2\right\},$$

and volume of portion of stone left for n^{th} man

$= \pi x^2 \cdot 2(y+c) - \frac{1}{3}\pi y(y^2 + 3x^2) = 2\pi c x^2 + \frac{2}{3}\pi\left\{3rx^2 - 2r^3 + 2(r^2 - x^2)^{\frac{3}{2}}\right\}$, by ($\alpha$),

Thus finally, we obtain as the equation for finding x

$$2n\left\{3(c+r)x^2 - 2r^3 + 2(r^2 - x^2)^{\frac{3}{2}}\right\} = a^3\left\{6b - m(m^2+3)a\right\},$$

where $$r = a\frac{1+m^2}{2m}.$$

4. If the stone be in the shape of a double concave paraboloidal lens, let p = parameter of generating parabola, then, since the volume of a paraboloid of revolution is half the containing cylinder, the equation to determine x in this case will be

$$\left(\text{since } y = \frac{x^2}{4p}\right) \quad \pi x^2 \cdot 2\left(\frac{x^2}{4p} + c\right) - \pi x^2 \cdot \frac{x^2}{4p} = \frac{1}{n}\left\{\pi a^2 \cdot 2b - \pi a^2(b-c)\right\};$$

or $$nx^2(x^2 + 8pc) = 4pa^2(b+c);$$

but $$a^2 = 4p(b-c) = 4pma, \quad \text{therefore } 4p = a/m;$$

thus we have finally the equation $nx^2(mx^2 + 2ac) = a^3(b+c)$.

8200. (By Rev. T. C. Simmons, M.A.)—Two straight lines are drawn at random in a given plane. Is the chance of their parallelism infinitesimal or zero?

Remarks by D. Biddle.

The best way of answering this question is perhaps the following:—Granting that the probability in the case before us is but slight, would it not, however, be less, if the direction and distance apart of the two lines, as well as the parallelism, were specifically mentioned? And would it not be still less, if the second line drawn were not necessarily in the same plane as the first? If four points be taken in a sphere, we can easily discover the following grades of probability:—1st, that the four points shall be in one plane; 2nd, that that plane shall be one previously fixed upon; 3rd, that two of the lines joining the points shall be parallel; 4th, that they shall also be parallel to a given line; 5th, that they shall be a given distance apart; 6th, that the quadrilateral formed shall be a parallelogram; 7th, a square.

Now, how can we hold that the 1st grade of probability here mentioned is *absolute zero*, when it is clear that, whilst none of the seven events is beyond the range of possibility, the probability of each succeeding grade is infinitely less than that preceding it? The probability that four points, taken at random in a sphere, shall lie in one plane, is infinitesimal—*not* zero; and by a parity of reasoning, the same can be said of the probability that two straight lines, drawn at random in one plane, shall be parallel. The probability as to the parallelism of three random lines is less than that of two; of four lines less than that of three; and so on, *ad infinitum*.

8161. (By Émile Vigarié.)—On donne un cercle O de diamètre AB, deux tangentes parallèles AP, BQ, une troisième tangente mobile PQ et un point fixe C sur le diamètre. On abaisse OD, OE respectivement perpendiculaires sur PC, QC; la droite DE coupe AB en F. Enfin on élève CG perpendiculaire à AB; cette droite coupe la circonférence en G. Démontrer que (1) le point F est fixe; et (2) la droite GF est la médiane antiparallèle du triangle GOC.

Solution by R. Knowles, B.A.; Emily Perrin; *and others.*

If $x^2+y^2=a^2$ be the equation to the circle, (x, y), $(b, 0)$ the coordinates of the point of contact, and of C respectively, the equations of PC, QC, OD, OE, are severally $y'(a-b)\,y = a(a-x')(x-b)$, $y'(a+b)\,y = a(a+x')(b-x)$, $a(a-x')\,y = -y'(a-b)$, $a(a+x')\,y = y'(a+b)$; hence we can find the coordinates of D and E, and the equation to DE, which is

$$y + \frac{ab(a-b)\,y'}{(a-b)^2(a+x') + a^2(a-x')} = \frac{2aby'(2a^2-b^2)}{4a^3b(x'-b)}\left(x - \frac{a^2b(a-x')}{(a-b)^2(a+x') + a^2(a-x')}\right),$$

and, putting $y = 0$, we find for F, $x = \frac{a^2b}{2a^2-b^2}$, $y = 0$; and since $\frac{a^2b}{2a^2-b^2}$, when a is greater than b, is less than b; therefore F lies between O and C.

The circle $x^2+y^2-bx-2ky = 0$, in which k is variable, will represent a circle of which OC is a chord, and will meet the lines OG and CG in points whose coordinates are respectively

$$\frac{b}{a^2}\left[b^2+2k\,(a^2-b^2)^{\frac{1}{2}}\right],\ \frac{(a^2-b^2)^{\frac{1}{2}}}{a^2}\left[b^2+2k\,(a^2-b^2)^{\frac{1}{2}}\right],\ (b,\ 2k),$$

and those of the mid-point of this line are

$$\frac{b}{2a^2}\left[a^2+b^2+2k\,(a^2-b^2)^{\frac{1}{2}}\right] \text{ and } \frac{4a^2k-2b^2k+b^2\,(a^2-b^2)^{\frac{1}{2}}}{2a^2},$$

and these points satisfy the equation to GF, viz.,

$$y = \frac{2a^2-b^2}{b\,(a^2-b^2)^{\frac{1}{2}}}\left(x-\frac{a^2b}{2a^2-b^2}\right);$$

hence GF is the median anti-parallel of the triangle GOC.

Otherwise:—Putting $2a^2x = a^2b+b^3+2bk\,(a^2-b^2)^{\frac{1}{2}}$,

and $$2a^2y = 4a^2k-2b^2k+b^2\,(a^2-b^2)^{\frac{1}{2}},$$

and eliminating k, we get the above equation to GF as the locus of the mid-points.

8278. (By Professor HUDSON, M.A.)—Prove that the locus of the point of intersection of normals to $y^2 = 4ax$ at the points where it is intersected by a tangent to the locus of the intersection of normals at right angles is $3375ay^2 = 16\,(x-5a)\,(x-50a)^2$.

Solution by Rev. D. THOMAS, M.A.; Rev. T. GALLIERS, M.A.; *and others.*

The three normals to $y^2 = 4ax$ through the point $(\xi,\ \eta)$ are given by

$$am^3-(\xi-2a)\,m-\eta = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1),$$

and the chord joining the extremities of two is

$$2x-(m_1+m_2)\,y+2am_1m_2 = 0,$$

or, from (1), $2m_3x+m_3^2y+2\eta = 0$, which touches $y^2-ax+3a^2 = 0$, the locus in the question; therefore

$$am_3^3-48am_3-16\eta = 0 \text{ and } am_3^3-(\xi-2a)\,m_3-\eta = 0, \text{ from (1).}$$

Hence, eliminating m_3, we get $15^3a\eta^2 = 16\,(\xi-5a)\,(\xi-50a)^2$.

8327. (By W. J. C. SHARP, M.A.)—If a conic touch the three lines drawn from the intersections of a given straight line with the sides of a triangle to the opposite angles, prove that the intersections of the other three tangents from the vertices of the triangle with the opposite sides are collinear.

Solution by Rev. T. Galliers, M.A.; J. O'Regan; *and others.*

Employing trilinear coordinates, let the equations of the given straight line and conic be $l\alpha + m\beta + n\gamma = 0$ .....(1),

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0 \text{ .....(2).}$$

Then (see Ferrers' *Tril. Co.*, 1st ed., Ch. iv., Art. 10) the equation of the two tangents to the conic drawn through A will be

$$W\beta^2 - 2U'\beta\gamma + V\gamma^2 = 0, \quad V = wu - v'^2, \quad W = uv - w'^2, \quad v'w' - uu' = U' \text{...(3).}$$

Now, by the question, one of these tangents is the straight line drawn from A to the point where (1) meets BC, that is, the straight line whose equation is $m\beta + n\gamma = 0$ .....(4).

Let the equation of the other tangent drawn through A be

$$\beta + \rho\gamma = 0 \text{ .....(5).}$$

Then, comparing equations (4) and (5) with equation (3), we have

$$\frac{n}{m}\rho = \frac{V}{W};$$

so that (5) may be written $\frac{\beta}{mV} + \frac{\gamma}{nW} = 0$ .....(6).

Similarly the other tangents drawn (in like manner) through B and C have for their equations respectively

$$\frac{\gamma}{nW} + \frac{\alpha}{lU} = 0, \quad \frac{\alpha}{lU} + \frac{\beta}{mV} = 0 \text{.....(7, 8);}$$

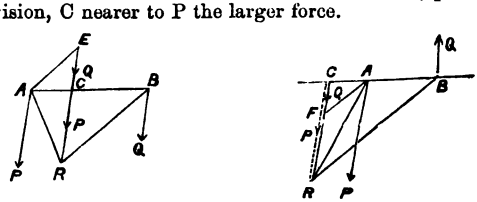
whence it follows that the points where (6), (7), (8) meet BC, CA, AB respectively lie on the straight line $\frac{\alpha}{lU} + \frac{\beta}{mV} + \frac{\gamma}{nW} = 0$.

[The property in the question is the polar reciprocal of the well-known theorem, that if the lines drawn from the opposite angles to three of the intersections of a conic with the sides of a triangle be concurrent, the other three are also concurrent.]

Proof of the Rule for the Composition of Two Parallel Forces.

By J. Walmsley, B.A.

Let A, B be points in lines of action of parallel forces P and Q respectively. Join AB and divide AB so that BC : CA = P : Q, putting, in the external division, C nearer to P the larger force.



Take BC, CA to represent P, Q respectively in *magnitude.*

Through C draw ER or CR parallel to directions of P, Q; make CE = CA = Q, CR = CB = P, and join EA, AR, RB.

Then we easily show that EA is parallel to BR. (Euc. Bk. I. or VI.)

Also CR in triangle ACR may represent P at A, and EC in ACE may represent Q at B. Therefore, by the aid of these triangles, we may resolve P at A into CA, AR at A; and Q at B into EA, AC at B.

But AC at B and CA at A equilibrate; therefore the original forces are equivalent to AR at A and EA at B, the latter of which must really act along BR or RB.

Both these forces, by the "principle of transmissibility," may be taken as acting at R.

Their resultant is therefore ER, the third side of the triangle EAR.

Hence the usual conclusions for like and unlike forces, with an implied graphic determination of resultant.

8219. (By Professor WOLSTENHOLME, M.A., Sc.D.)—If $y^5 - 5y^3 + 5y = 2x$, prove (1) that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \frac{1}{25}y = 0$; and (2), by means of this equation, obtain the five expansions of y in series of positive ascending powers of x.

Solution by B. HANUMANTA RAU, B.A.; J. BRILL, B.A.; *and others.*

1. Put $y = 2\cos\theta$, then $x = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta = \cos 5\theta$,

therefore
$$\frac{dy}{dx} = 2\sin\theta \,/\, 5\sin 5\theta,$$

$$\frac{d^2y}{dx^2} = -\frac{2}{5}\left(\frac{\cos\theta}{\sin 5\theta} - \frac{5\sin\theta\cos 5\theta}{\sin^2 5\theta}\right) \Big/ 5\sin 5\theta;$$

therefore
$$(1-x^2)\frac{d^2y}{dx^2} = -\frac{2\cos\theta}{25} + \frac{2\sin\theta\cos 5\theta}{5\sin 5\theta} = -\frac{1}{25}y + x\frac{dy}{dx},$$

or
$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + \tfrac{1}{25}y = 0.$$

2. The integral of this equation is

$$y = c_1 e^{\frac{1}{5}\sin^{-1}x} + c_2 e^{-\frac{1}{5}\sin^{-1}x},$$

where c_1, c_2 are determined from the equations

$$c_1 + c_2 = y_0 \text{ and } c_1 - c_2 = \left(\frac{dy}{dx_0}\right).$$

When $x = 0$, $y_0 = 0$ or $y_0^4 - 5y_0^2 + 5 = 0$,

therefore $y_0^2 = \frac{5}{2} \pm \frac{1}{2}\sqrt{5} = 2(1-\sin 18°)$ or $2(1+\cos 36°)$,

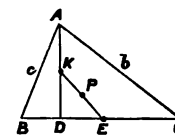
therefore $y_0 = \pm 2\sin 36°$ or $\pm 2\cos 18°$.

Again, $dy_0 \,/\, dx = \frac{1}{5} \,/\, (y_0^4 - 3y_0^2 + 1) = -\frac{2}{5}\sin 18°$ or $\frac{2}{5}\cos 36°$.

8167. (By the EDITOR.)—Prove that the Symmedian point of any triangle—viz., the cointersection of the lines joining the mid-point of each perpendicular with the mid-point of the corresponding side—may be regarded as the centre of inertia of three masses proportional to a^2, b^2, c^2, placed at the angular points of the triangle.

Solution by Prof. MATHEWS, M.A.; *and* Prof. IGNACIO BEYENS.

1. Let AD be the perpendicular from A on BC; K,E the middle points of AD, BC; then it can be shown that KE passes through the Symmedian point P. Suppose the mass at B to be replaced by two masses proportional to $\frac{1}{2}(a^2+b^2-c^2)$, and $\frac{1}{2}(b^2+c^2-a^2)$, (sum $= b^2$); and suppose the mass at C to be replaced by two masses proportional to $\frac{1}{2}(a^2+c^2-b^2)$, and $\frac{1}{2}(b^2+c^2-a^2)$, (sum $= c^2$); and let us consider what is the centre of inertia of the new system thus obtained. The mass $\frac{1}{2}(a^2+b^2-c^2)$ at B, and $\frac{1}{2}(a^2+c^2-b^2)$ at C is equivalent to a mass



$$\tfrac{1}{2}(a^2+b^2-c^2)+\tfrac{1}{2}(a^2+c^2-b^2) = a^2 \text{ at D, for}$$

$$BD : CD = c\cos B : b\cos C = 2ac\cos B : 2ab\cos C = a^2+c^2-b^2 : a^2+b^2-c^2,$$

and a mass a^2 at D with a^2 at A, is equivalent to a mass $2a^2$ at K. Similarly, the equal masses $\frac{1}{2}(b^2+c^2-a^2)$ at B and C are equivalent to a mass $b^2+c^2-a^2$ at E, the middle point of BC. Hence KE must pass through P, the centre of inertia of the system, and

$$KP : PE = b^2+c^2-a^2 : 2a^2.$$

[If, through P, EPF′, FPD′, DPE′ be drawn parallel respectively to BC, CA, AB (see the Figure on p. 102, of Vol. XL.), and if AP meet BC in X, we have

$$BX : XC = F'P : PE = F'F : FA = a^2 : b^2;$$

whence X is the centroid of b^2 at B, and c^2 at C.

Again, we have also

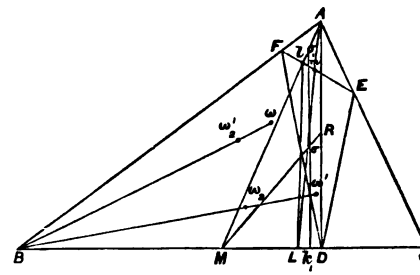
$$AP : PX = AE : EC = b^2+c^2 : a^2,$$

whence P is the centroid of b^2+c^2 at X, and a^2 at A, which is the required result. See Quests. 8326, 8388.]

8287. (By R. TUCKER, M.A.)—DEF is the pedal triangle of ABC, whose Brocard-points are ω, ω', and Lemoine-point σ: the Brocard-points and Lemoine-points of AFE, BDF, CED are respectively ω_1, ω_1', σ_1, and so on; prove that (1) ω_1, ω_1'; ω_2, ω_2'; ω_3, ω_3' are concyclic with ω, ω' (*i.e.*, the three circles are coaxal); (2) if ρ', ρ_1', ρ_2', ρ_3' are the Brocard-radii of ABC and of AFE, &c., then $\rho_1'^2+\rho_2'^2+\rho_3'^2 = \rho'^2(1-2\cos A\cos B\cos C)$; (3) the following construction will give the Lemoine-point of AFE, &c.:—Produce the perpendicular $\kappa\sigma$ (on BC) to a point σ_1, such that $\kappa\sigma_1 = 2\kappa\sigma$; this is the point sought; (4) σ is the mean centre (or centroid) of σ_1, σ_2, σ_3.

Solution by Rev. T. C. SIMMONS, M.A.

1. Since $\angle BDF = BAC$ and $\angle BFD = BCA$, the triangles BAC, BFD are inversely similar, and B is the centre of similitude. Therefore $B\omega_2$ makes the same angle with BD as $B\omega$ makes with BA, *i.e.*, $B\omega_2\omega'$ is a straight line, and likewise $B\omega_2'\omega$.



Moreover,

$B\omega_2 : B\omega_2' = B\omega : B\omega'$,

whence it follows that ω, ω', ω_2, ω_2' are concyclic, and similarly the Brocard-points of AEF, CDE.

2. Since the ratio of similitude of AEF, ABC is AF : AC or cos A, it follows that, if ρ denote radius of *any* circle belonging to ABC (circumcircle, incircle, excircle, Brocard-circle, "T.R." circle, or any other), and ρ_1, ρ_2, ρ_3 corresponding radii for the triangles AEF, BFD, CED, then $\rho_1 = \rho \cos A$, $\rho_2 = \rho \cos B$, $\rho_3 = \rho \cos C$; whence

$$\rho_1^2 + \rho_2^2 + \rho_3^2 = \rho^2 (\cos^2 A + \cos^2 B + \cos^2 C) = \rho^2 (1 - 2 \cos A \cos B \cos C),$$

a relation which, being true in general, follows for the particular case of the Brocard-radius.

3. Since A is the centre of similitude of the triangles AFE, ABC, and $A\sigma$ is known to bisect FE in m, therefore (σ_1 being the Lemoine-point of AFE) $A\sigma_1$ produced bisects BC in M; whence $M\sigma$ produced bisects AD in R. Again, if $A\sigma$ meets BC in L and $A\sigma_1$M meets EF in l, we have $A\sigma : \sigma L :: A\sigma_1 : \sigma_1 l$, showing that $\sigma\sigma_1$ is parallel to Ll. But, since M, m are corresponding points, also L, l; it follows that $Al : Am : AL : AM$, so that M, l, m, L are concyclic, whence $\angle MLl = Mml$. Moreover, since M is the centre of the semicircle BFEC, the triangle MFE is isosceles, showing that MmF is a right angle. Hence Ml is also a right angle, showing that Ll, which has been proved to be parallel to $\sigma\sigma_1$, is also parallel to $\kappa_1\sigma$ the perpendicular from σ on BC. Thus $\kappa_1\sigma\sigma_1$ is a straight line, and therefore $\kappa_1\sigma : \sigma\sigma_1 :: DR : RA$, or $\kappa_1\sigma = \sigma\sigma_1$.

4. Since $\sigma\kappa_1 = \sigma\sigma_1$, $\sigma\kappa_2 = \sigma\sigma_2$, $\sigma\kappa_3 = \sigma\sigma_3$, σ is the centre of similitude of the similar triangles $\kappa_1\kappa_2\kappa_3$ and $\sigma_1\sigma_2\sigma_3$. But it is known to be the centroid of the former triangle; therefore it is also the centroid of the latter, and consequently of the points σ_1, σ_2, σ_3.

8230. (By PIERS C. WARD, M.A.) — Show that the rationalization of $a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} + d^{\frac{1}{3}} = 0$ is equivalent to the eliminant of x from

$$(x - a - b)^3 - 27abx = 0, \quad \text{and} \quad (x + c + d)^3 - 27cdx = 0.$$

Solution by B. HANUMANTA RAU, M.A.; J. BRILL, M.A.; *and others.*

$a^{\frac{1}{3}} + b^{\frac{1}{3}} + c^{\frac{1}{3}} + d^{\frac{1}{3}} = 0$ is obtained by eliminating x from the equations

$$x^{\frac{1}{3}} - a^{\frac{1}{3}} - b^{\frac{1}{3}} = 0, \text{ and } x^{\frac{1}{3}} + c^{\frac{1}{3}} + d^{\frac{1}{3}} = 0,$$

or from $$x - a - b = 3a^{\frac{1}{3}}b^{\frac{1}{3}}x^{\frac{1}{3}}, \quad x + c + d = 3c^{\frac{1}{3}}d^{\frac{1}{3}}x^{\frac{1}{3}},$$

or from $$(x-a-b)^3 = 27abx \text{ and } (x+c+d)^3 = 27cdx.$$

7908. (By R. KNOWLES, B.A.)—Tangents are drawn from a point T (h, k) to meet the ellipse $a^2y^2 + b^2x^2 = a^2b^2$, centre C, in P and Q; prove that (1) $CP^2 \sim CQ^2 = [4a^2b^2(a^2-b^2)\,hk\,(a^2k^2 + b^2h^2 - a^2b^2)]^{\frac{1}{2}} / (a^2k^2 + b^2h^2)^2$; (2) if T be on either axis, CP and CQ are equal.

Solution by G. G. STORR, B.A.; J. O'REGAN; *and others.*

The equations to PQ and ellipse are $b^2hx + a^2ky = a^2b^2$, $a^2b^2 = b^2x^2 + a^2y^2$, therefore equation to CP and CQ is $(b^2hx + a^2ky)^2 = a^2b^2(b^2x^2 + a^2y^2)$, therefore

$$(y/x)^2(a^4k^2 - a^4b^2) + 2\,.\,y/x\,.\,a^2b^2hk + b^4h^2 - a^2b^4 = 0.$$

Let m_1, m_2 be the roots of this equation, and let the equations to CP, CQ be $y = m_1x$, $y = m_2x$; then we find

$$x = \frac{ab}{(a^2m_1^2 + b^2)^{\frac{1}{2}}}, \quad y = \frac{abm_1}{(a^2m_1^2 + b^2)^{\frac{1}{2}}},$$

coordinates of P; therefore

$$CP^2 = \frac{a^2b^2(1+m_1^2)}{a^2m_1^2 + b^2}, \text{ and } CQ^2 = \frac{a^2b^2(1+m_2^2)}{a^2m_2^2 + b^2},$$

therefore $$CP^2 - CQ^2 = a^2b^2\left\{\frac{1+m_1^2}{a^2m_1^2 + b^2} - \frac{1+m_2^2}{a^2m_2^2 + b^2}\right\}$$

$$= a^2b^2\,\frac{(a^2-b^2)(m_2+m_1)\left\{(m_2+m_1)^2 - 4m_1m_2\right\}^{\frac{1}{2}}}{a^4(m_1m_2)^2 + a^2b^2\left\{(m_1+m_2)^2 - 2m_1m_2\right\} + b^4}.$$

Substituting for $m_1 + m_2$, $-\frac{2a^2b^2hk}{a^4(k^2-b^2)}$, and for m_1m_2, $\frac{b^4(h^2-a^2)}{a^4(k^2-b^2)}$, and simplifying, we have the stated result.

8353. (By Professor BOOTH, M.A.)—Eliminate (1) x, y, z from the equations $ax^2 + 2hxy + by^2 = 0$, $by^2 + 2fyz + cz^2 = 0$, $cz^2 + 2gzx + ax^2 = 0$, and give (2) a geometrical interpretation of the result.

Solution by R. GRAHAM, M.A.; SARAH MARKS; *and others.*

1. Calling $x\sqrt{a}$, $y\sqrt{b}$, $z\sqrt{c}$, $\frac{f}{\sqrt{bc}}$, $\frac{g}{\sqrt{ca}}$, $\frac{h}{\sqrt{ab}}$, respectively, α, β, γ, λ, μ, ν the given equations become

$$\beta^2+\gamma^2+2\lambda\beta\gamma=0, \quad \gamma^2+\alpha^2+2\mu\gamma\alpha=0, \quad \alpha^2+\beta^2+2\nu\alpha\beta=0.$$

Now, we know that

$$a^2(\beta^2+\gamma^2)^2+\beta^2(\gamma^2+\alpha^2)^2+\gamma^2(\alpha^2+\beta^2)^2-4a^2\beta^2\gamma^2\equiv(\beta^2+\gamma^2)(\gamma^2+\alpha^2)(\alpha^2+\beta^2),$$

therefore $a^2(4\lambda^2\beta^2\gamma^2)+\beta^2(4\mu^2\gamma^2\alpha^2)+\gamma^2(4\nu^2\alpha^2\beta^2)-4a^2\beta^2\gamma^2$

$$=(-2\lambda\beta\gamma)(-2\mu\gamma\alpha)(-2\nu\alpha\beta),$$

therefore $\lambda^2+\mu^2+\nu^2-1=-2\lambda\mu\nu$, or $abc-2fgh-af^2-bg^2-ch^2=0$.

2. The three given equations are those of the lines joining the vertices of the triangle of reference with the points where the opposite sides meet the conic $(abc/gh)(xyz)^2=0$, hence the eliminant of the given equations expresses that these six lines pass three and three through two common points.

8265. (By W. J. GREENSTREET, B.A.)—If P, Q, R are points on an ellipse, P′, Q′, R′ the ends of corresponding conjugate diameters, and circles, radii r, r', are drawn circumscribing triangles PQR, P′Q′R′; show that $r^2+r'^2=\rho^2+\rho'^2+a^2+b^2$, where ρ, ρ' are the distances of the centres of these circles from the centre of the ellipse.

Solution by A. GORDON; W. E. HEAL, M.A.; *and others.*

Let the eccentric angles of P, Q, R be α, β, γ, and hence of P′, Q′, R′ are $\alpha+\frac{1}{2}\pi$, $\beta+\frac{1}{2}\pi$, $\gamma+\frac{1}{2}\pi$; the equation of the circle about three given points α, β, γ (P, Q, R) [SALMON's *Conics*, p. 209, Ex. 8] is

$$x^2+y^2-\frac{2(a^2-b^2)}{a}x\,.\cos\tfrac{1}{2}(\alpha+\beta)\cos\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\gamma+\alpha)$$

$$-\frac{2(b^2-a^2)}{b}y\,.\sin\tfrac{1}{2}(\alpha+\beta)\sin\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\gamma+\alpha)$$

$$=\tfrac{1}{2}(a^2+b^2)-\tfrac{1}{2}(a^2-b^2)\{\cos(\alpha+\beta)+\ldots\}.$$

Similarly about P′Q′R′ obtained by changing α into $\frac{1}{2}\pi+\alpha$, &c. Hence

$$r^2=\tfrac{1}{2}(a^2+b^2)+\rho^2-\tfrac{1}{2}(a^2-b^2)[\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)],$$

$$r'^2=\tfrac{1}{2}(a^2+b^2)+\rho'^2+\tfrac{1}{2}(a^2-b^2)[\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)],$$

therefore $r^2+r'^2=a^2+b^2+\rho^2+\rho'^2$.

8348. (By Professor CATALAN.)—Un triangle ABC étant inscrit à un cercle, soient EDF le diamètre perpendiculaire au coté AB, et G la projection de C sur EF; la demi-somme des cotés AC, BC est moyenne proportionnelle entre les segments DF, GE; leur demi-différence est moyenne proportionnelle entre les segments DE, FG.

Solutions by (1) Prof. IGNACIO BEYEUS; (2) *the* PROPOSER; *and others.*

1. We have $\frac{1}{2}AC = R \sin B$, $\frac{1}{2}BC = R \sin A$,

$\frac{1}{2}(AC + BC) = R(\sin B + \sin A)$

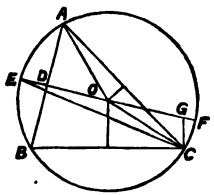
$= 2R \sin \frac{1}{2}(B + A) \cos \frac{1}{2}(B - A)$;

$EG : EF = EG^2 : EC^2 = \cos^2 GEC$

$= \cos^2 \frac{1}{2} GOC = \cos^2 \frac{1}{2}(B - A)$,

$DF = R + OD = R(1 + \cos DOA)$

$= R(1 + \cos C) = 2R \sin^2 \frac{1}{2}(B + A)$.



Squaring and substituting, we have

$\frac{1}{4}(AC + BC)^2 = EG \cdot DF$;

so also $\frac{1}{2}(AC - BC) = 2R \cos \frac{1}{2}(B + A) \sin \frac{1}{2}(B - A)$,

$GF : EF = GF^2 : FC^2 = \sin^2 \frac{1}{2}(B - A)$,

$DE = R(1 - \cos DOA) = 2R \cos^2 \frac{1}{2}(B + A)$.

Squaring and substituting, $\frac{1}{4}(AC - BC)^2 = DE \cdot FG$.

2. *Otherwise*:—On sait que

$$(AC)^2 + (BC)^2 = 2(AD)^2 + 2(CD)^2, \quad AC \cdot BC = EF \cdot DG.$$

Donc $$(AC \pm BC)^2 = 2\left[(AD)^2 + (CD)^2 \pm EF \cdot DG\right].$$

D'un autre côté,

$$(AD)^2 = DE \cdot DF, \quad (CD)^2 = (DG)^2 + (CG)^2 = (DG)^2 + GE \cdot GF.$$

Ainsi $$(AC \pm BC)^2 = 2\left[DE \cdot DF + (DG)^2 + GE \cdot GF \pm EF \cdot GD\right].$$

Soient, pour abréger, a, b, c les segments FG, GD, DE. La quantité entre parenthèses égale $b^2 + (a+b)c + (b+c)a \pm (a+b+c)b$. Selon que l'on adopte le signe + ou le signe −, elle se réduit à

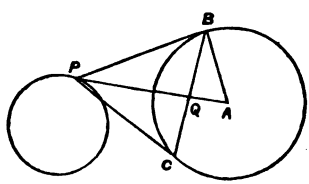
$$2(b+a)(b+c) = 2 \cdot DF \cdot GE, \text{ ou à } 2ac = 2 \cdot GF \cdot DE.$$

8112 & 8147. (By R. KNOWLES and the Rev. J. J. MILNE, M.A.)—Given two circles which do not intersect; from any point on the one, tangents are drawn to the other: prove that the envelope of the chord of contact and the locus of its middle point are respectively a hyperbola and its auxiliary circle.

Solution by Rev. J. J. MILNE, M.A.; A. GORDON; *and others.*

Join AP. Then AP bisects BC at right angles; and ABP is a right angle, therefore $AQ \cdot AP = AB^2 = \text{const.}$

Now A is a fixed point and locus of P a circle; therefore locus of Q is a circle, A being the centre of similitude of the two. And, QB being at right angles to AQ, BC envelopes a hyperbola of which A is one focus, and the locus of Q the auxiliary circle.



8193. (By Professor HUDSON, M.A.)—A cone, with vertex up and closed with a plane horizontal base, contains $\frac{1}{16}$ of its volume of water, and rotates with uniform angular velocity about a vertical axis: show that, when the angular velocity is such that the free surface touches the base of the cone, it bisects the slant side.

Solution by A. GORDON; Rev. T. GALLIERS, M.A.; *and others.*

Assuming that there is no impressed force but gravity, and that the fluid is incompressible, and rotating about a vertical axis, the surfaces of equal pressure are given by $p/\rho = \frac{1}{2}\omega^2 r^2 - gz$, where the origin is taken at the lowest point of the free surface (in this case at the centre of the base), and the axis of α is vertically upwards. Let h be the height of the cone, z its semi-vertical angle, the volume of water

$$= \tfrac{1}{16} \text{ cone} = \tfrac{1}{16} \cdot \tfrac{1}{3}\pi h^3 \tan^2 \alpha.$$

Let y be the altitude of water level when revolving

$$\int_0^y \pi \left[(h-z)^2 \tan^2 \alpha - \frac{2gz}{\omega^2} \right] dz = \text{volume of water} = \tfrac{1}{16} \cdot \tfrac{1}{3}\pi h^3 \tan^2 \alpha \ \ldots \ (1),$$

also $(y-h)^2 \tan^2 \alpha = \dfrac{2gy}{\omega^2}$ ..... (2). Put $\lambda = \dfrac{2g}{\omega^2} \cot^2 \alpha$ ..... (3).

Substituting for ω^2 from (3) in (1) and (2), we have

$$(y-h)^2 = \lambda y, \quad \text{and} \quad 16\left[(y-h)^3 + h^3\right] - 24\lambda y^2 = 11h^3,$$

or, by reducing, $8y^3 - 24h^2 y + 11h^3 = 0$, or $(2y-h)(4y^2 + 2hy - 11h^2) = 0$, the only possible value of y (*i.e.*, between h and 0) is $y = \frac{1}{2}h$.

8273. (By J. BRILL, M.A.)—Prove that the locus of the centroid of the points in which a variable tangent to a parabola is intersected by n fixed tangents is also a tangent of the parabola, and its point of contact lies on the line joining the centroid of the $\frac{1}{2}n(n-1)$ points of intersection of the fixed tangents to the centroid of their points of contact, and divides that line in the ratio of $1 : n-1$.

Solution by Professor SWAMINATHA AIYAR, B.A.

Let $m_1, m_2, m_3 \ldots\ldots m_n$ be the tangents of the angles made by the n fixed tangents with the axis of x.

The point of contact of any tangent (m_1) is $\left(\dfrac{a}{m_1^2}, \dfrac{2a}{m_1}\right)$.

The point of intersection of any two tangents (m_1, m_2) is $\left(\dfrac{a}{m_1 m_2}, \dfrac{a}{m_1} + \dfrac{a}{m_2}\right)$.

The centroid of the points of contact is the point $\left(\dfrac{a}{n} \Sigma \dfrac{1}{m_1^2}, \dfrac{2a}{n} \Sigma \dfrac{1}{m_1}\right)$.

The centroid of the points of intersection of the tangents is the point whose coordinates are $\dfrac{2a}{n(n-1)} \Sigma \dfrac{1}{m_1 m_2}$ and $\dfrac{2a}{n} \Sigma \dfrac{1}{m_1}$.

The straight line joining these two centroids is $y = \frac{2a}{n} \Sigma \frac{1}{m_1}$.

This line cuts the parabola $y^2 = 4ax$ at the point whose abscissa is $\frac{a}{n^2} \Sigma \frac{1}{m_1^2} + \frac{2a}{n^2} \Sigma \frac{1}{m_1 m_2}$; and is therefore cut in the ratio of 1 to $n-1$ at this point.

Again, the point of intersection of any variable tangent (m) with one of the fixed tangents (m_1) is $\left(\frac{a}{m m_1}, \frac{a}{m} + \frac{a}{m_1}\right)$.

The centroid of the points of intersection of this variable tangent with the n fixed tangents is the point $\left(\frac{a}{mn} \Sigma \frac{1}{m_1}, \frac{a}{m} + \frac{a}{n} \Sigma \frac{1}{m_1}\right)$.

This centroid evidently lies on the line

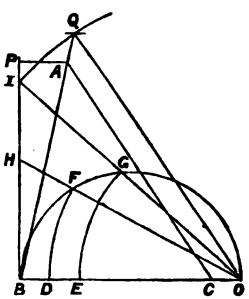
$$\left(y - \frac{a}{n} \Sigma \frac{1}{m_1}\right) \Big/ a = x \Big/ \left(\frac{a}{n} \Sigma \frac{1}{m_1}\right),$$

that is, on the line $y = x \cdot 1 \Big/ \left(\frac{1}{n} \Sigma \frac{1}{m_1}\right) + a \cdot \frac{1}{n} \Sigma \frac{1}{m_1}$,

which line again is evidently a tangent to the parabola, the ordinate of the point of contact being $\frac{2a}{n} \Sigma \frac{1}{m_1}$.

8192. (By Professor NILKANTHA SARKAR, M.A.)—Given the perpendiculars from the angular points of a triangle on the opposite sides, construct the triangle.

Solution by D. BIDDLE; J. O'REGAN; *and others.*



Let OB, OD, OE = the three perpendiculars. On OB describe a semicircle, and draw BP at right angles and equal to OB. With centre O, and radii OD, OE respectively, draw the arcs DF, EG, to cut the semicircle; and produce OF, OG to H, I, on the perpendicular. With centre O, and radius OI, describe the arc IQ, and make BQ = OH. Join BQ, OQ, and draw PA parallel to BO. Finally, draw AC parallel to OQ. ABC is the triangle required.

For it is evident that the sides of the required triangle must be inversely as the perpendiculars upon them from the opposite angles. But OF : OB = OB : OH, and OG : OB = OB : OI; therefore OB, BQ, OQ are inversely as OB, OD, OE. Consequently OBQ is similar to the required triangle. But ABC is similar to OBQ, and the perpendicular on its shortest side = OB. Hence, &c.

8351. (By Professor HUDSON, M.A.)—Prove that the two generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ through the point (x, y, z) will meet the principal elliptic section at the ends of diameters at right angles if

$$\frac{x^2+y^2}{a^2+b^2}=\frac{z^2}{c^2}.$$

Solution by R. F. DAVIS, M.A.; EMILY PERRIN; *and others.*

The orthogonal projections of these generators upon the plane (xy) are tangents to the principal elliptic section at the points through which the generators pass. Now tangents to an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the extremities of perpendicular diameters intersect on the ellipse

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}.$$

This equation may be written in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=\frac{x^2+y^2}{a^2+b^2}, \text{ whence, \&c.}$$

8247. (By Professor HUDSON, M.A.)—If $S_r=1^r+2^r+3^r+\ldots+n^r$, prove that $\frac{(n+1)^{r+1}-n^{r+1}-(n+1)+(-)^r n}{(r+1)\,r}=S_{r-1}+\frac{(r-1)(r-2)}{3.4}S_{r-3}+\frac{(r-1)(r-2)(r-3)(r-4)}{3.4.5.6}S_{r-5}+\ldots$, ending with the term involving S_2 or S_1 according as r is odd or even.

Solution by Professor SWAMINATHA AIYAR, B.A.

$$(e^x+e^{2x}+e^{3x}+\ldots+e^{nx})(e^x+e^{-x}-2)=e^{(n+1)x}-e^{nx}-e^x+1\ldots\ldots (1),$$

$$e^x+e^{2x}+e^{3x}\ldots e^{nx}=S_0+S_1x+\frac{S_2}{2!}x^2+\frac{S_3}{3!}x^3\ldots\ldots,$$

$$e^x+e^{-x}-2=\frac{2x^2}{2!}+\frac{2x^4}{4!}+\frac{2x^6}{6!}\ldots\ldots$$

Equating the coefficients of x^{r+1} in the expansions of the two sides of (1), we have $\frac{2S_{r-1}}{r-1!\;2!}+\frac{2S_{r-3}}{r-3!\;4!}+\frac{2S_{r-5}}{r-5!\;5!}\ldots\ldots=\frac{(n+1)^{r+1}-n^{r+1}+1}{r+1!}$; from which the required relation follows very easily.

8251. (By Professor MATZ, M.A.)—A candidate is examined in three papers, to each of which m marks are assigned as a maximum; show that, if his total on the three papers is $2m$, there are $\frac{1}{2}(m+1)(m+2)$ ways in which this may occur.

Solution by C. MORGAN, B.A.; W. J. GREENSTREET, B.A.; *and others.*

Let the candidate obtain 0 marks in the third paper (say), then there is obviously only 1 way in which he may obtain $2m$ marks in the other two papers (for he must get full marks in them). Let candidate obtain 1 mark in the third paper, then he must have only 1 mark taken off in his other two papers, which can be done in 2 ways, viz., by taking 1 off either paper. Let candidate obtain 2 marks in the third paper, then he must have 2 taken off the other two papers, which can be done in 3 ways, viz., by taking 2 off either, or 1 off both. Similarly, if candidate obtain 3 marks in the third paper, there will be 4 ways in which these marks may be taken off the other two papers. Hence the total number of ways in which he may obtain $2m$ marks is

$$1+2+3+\ldots+(m+1)=\tfrac{1}{2}(m+2)(m+1).$$

8159. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Two circles with centres A, B have radii a, b, and the distance between their centres c, and $a > b$; a straight line moves so that the segments cut from it by the two circles have a given ratio $\lambda : 1$; prove that (1) the envelope will be an ellipse whose major axis lies along AB if λ be < 1 or $> a/b$; and an hyperbola whose transverse axis is along AB if $\lambda > 1$ and $< a/b$; and that, if S, S′ be the foci of the envelope, $AS' : AS = S'B : BS = \lambda+1 : \lambda-1$, so that S, S′ divide AB harmonically, and if C be the centre of the envelope $CA = \lambda . CS$; (2) when $\lambda = 1$, the envelope is a parabola whose focus bisects AB, whose tangent at the vertex is the radical axis of the two circles, and latus rectum is $2(a^2-b^2)/c$; (3) when $\lambda = a/b$, the envelope reduces to a point pair, the centres of similarity,—when λ is given by the equation $(a^2-\lambda^2b^2)(\lambda^2-1) = \lambda^2c^2$, which necessitates $a > b+c$, there is only one straight line which satisfies the condition, and the envelope reduces to this straight line $x = \lambda^2c/\lambda^2-1$; (4) if we assume λ a pure imaginary $\mu\sqrt{(-1)}$, we get for the envelope an ellipse or hyperbola whose conjugate axis lies along the line of centres, the envelope can never become a parabola or a point pair, but may reduce to a single straight line when

$$\mu^2c^2 = (a^2+\mu^2b^2)(\mu^2+1), \quad (c > a+b);$$

and this equation gives the maximum and minimum values of μ, but they cannot be considered geometrical.

Solution by Professor MATHEWS, M.A.

1. Let the equation of PQRS be

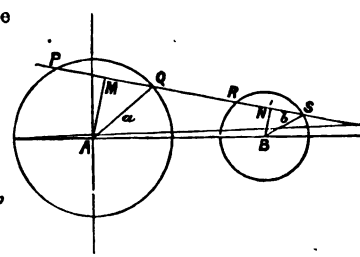
$$lx+my+n=0;$$

take A for origin, AB for axis of x; then

$$AM = \frac{n}{(l^2+m^2)^{\frac{1}{2}}}, \quad BN = \frac{lc+n}{(l^2+m^2)^{\frac{1}{2}}};$$

and, since $MQ^2 = \lambda^2 . NS^2$, *i.e.*, $AQ^2-AM^2 = \lambda^2(BS^2-BN^2)$, we have

$$a^2-\frac{n^2}{l^2+m^2} = \lambda^2\left\{b^2-\frac{(lc+n)^2}{l^2+m^2}\right\},$$



or $(a^2-\lambda^2b^2+\lambda^2c^2)\,l^2+(a^2-b^2\lambda^2)\,m^2+(\lambda^2-1)\,n^2+2\lambda^2c\,.\,nl=0.$

2. Calculating the reciprocal coefficients, these are

$$A=(a^2-b^2\lambda^2)(\lambda^2-1),\quad B=(a^2-b^2\lambda^2+c^2\lambda^2)(\lambda^2-1)-\lambda^4c^2$$
$$=(a^2-b^2\lambda^2)(\lambda^2-1)-c^2\lambda^2,$$
$$C=(a^2-b^2\lambda^2)(a^2-b^2\lambda^2+c^2\lambda^2),\quad F=H=0,\quad G=-\lambda^2c\,.\,(a^2-b^2\lambda^2).$$

Thus the envelope is the conic

$$(a^2-\lambda^2b^2)(\lambda^2-1)\,.\,x^2+\left\{(a^2-\lambda^2b^2)(\lambda^2-1)-\lambda^2c^2\right\}y^2-2\lambda^2c\,(a^2-\lambda^2b^2)\,x$$
$$+(a^2-\lambda^2b^2)(a^2-\lambda^2b^2+\lambda^2c^2)=0.$$

3. This will be an ellipse or hyperbola according as

$$(a^2-\lambda^2b^2)(\lambda^2-1)\left\{(a^2-\lambda^2b^2)(\lambda^2-1)-\lambda^2c^2\right\}$$

is positive or negative.

The critical cases are (i.) $a^2-\lambda^2b^2=0$, or, $\lambda=a/b$.

The equation of the conic reduces to $-(a^2c^2/b^2)\,y^2=0$,

i.e., the line AB counting twice.

The centres of similitude are special points on this line,—in fact, AB appears as the limit of a very flat ellipse and a very flat hyperbola, having double contact at the centres of similitude.

(ii.) $\lambda=1$ the conic is $-c^2y^2-2c\,(a^2-b^2)\,x+(a^2-b^2)(a^2-b^2+c^2)=0$,

a parabola, etc., as stated.

4. $(a^2-\lambda^2b^2)(\lambda^2-1)-\lambda^2c^2=0$: the conic is

$$(a^2-\lambda^2b^2)(\lambda^2-1)\,x^2-2\lambda^2c\,(a^2-\lambda^2b^2)\,x+(a^2-\lambda^2b^2)(a^2-\lambda^2b^2+\lambda^2c^2)=0,$$

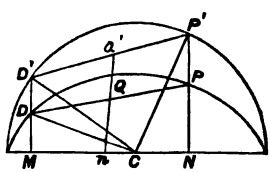
or $$(\lambda^2-1)\,x^2-2\lambda^2cx+(a^2-\lambda^2b^2)\,\lambda^2=0,$$

which may be written $(\lambda^2-1)\left\{x-(\lambda^2c/\lambda^2-1)\right\}^2=0$;

that is, the straight line $x=\lambda^2c/\lambda^2-1$ counted twice over.

8183. (By R. KNOWLES, B.A., L.C.P.) — CP, CD are conjugate radii of the ellipse $b^2x^2+a^2y^2=a^2b^2$; show that PD touches at its mid-point the ellipse $2b^2x^2+2a^2y^2=a^2b^2$.

Solution by Rev. J. J. MILNE, M.A.; Rev. T. GALLIERS, M.A.; *and others.*



Let Q be the middle point of PD, P′, Q′, D′ the points corresponding to P, Q, D. Then, since P′CD′ is a right angle, P′D′ touches at Q′ a concentric circle, of radius $\frac{1}{2}a\sqrt{2}$. Now $Qn=\frac{1}{2}(PN+DM)$, $Q'n=\frac{1}{2}(P'N+D'M)$; hence PD touches at Q the concentric ellipse

$$\frac{x^2}{(\frac{1}{2}a\sqrt{2})^2}+\frac{y^2}{(\frac{1}{2}b\sqrt{2})^2}=1.$$

8233. (By R. TUCKER, M.A.)—P is a point on an ellipse; PM, PN, perpendiculars on the equiconjugate diameters, are produced to meet their productions in M′, N′; the normal at P (which bisects MN) cuts MN in K and M′N′ in K′: find the loci of K, K′.

Solution by R. KNOWLES, B.A.; E. RUTTER; *and others.*

Let (x_1, y_1) be the coordinates of a point P on the ellipse $b^2x^2 + a^2y^2 = a^2b^2$, the equations to the equiconjugate diameters to PM and PN are respectively $y = \frac{b}{a}x$, $y = -\frac{b}{a}x$, $y - y_1 = -\frac{a}{b}(x - x_1)$, $y - y_1 = \frac{a}{b}(x - x_1)$,

therefore the coordinates of M and N are

$$\frac{a}{a^2+b^2}(ax_1 + by_1), \frac{b}{a^2+b^2}(ax_1 + by_1); \quad \frac{a}{a^2+b^2}(ax_1 - by_1), \frac{-b}{a^2+b^2}(ax_1 - by_1),$$

and the coordinates of the mid-point of MN are $\frac{a^2x_1}{a^2+b^2}$, $\frac{b_1}{a^2+b^2}$, and this point satisfies the equation to the normal at P, $a^2y_1x - b^2x_1y = (a^2 - b^2)x_1y_1$, and its locus is the result of the elimination of x_1y_1 from the equations $(a^2 + b^2)x = a^2x_1$, $(a^2 + b^2)y = b^2y_1$, and the curve, and is therefore the ellipse $b^2x^2 + a^2y^2 = a^6b^6 / (a^2 + b^2)^2$.

Similarly the coordinates of M′, N′ are

$$\frac{a(ax_1 + by_1)}{a^2 - b^2}, \frac{-b}{a^2 - b^2}(ax_1 + by_1); \quad \frac{a}{a^2 - b^2}(ax_1 - by_1), \frac{b(ax_1 - by_1)}{a^2 - b^2},$$

and the equation to M′N′ is

$$(a^2 - b^2)x_1x + (a^2 - b^2)y_1y = a^2x_1^2 - b^2y_1^2,$$

and is therefore at right angles to CP (C being the centre of the ellipse), and the coordinates of the intersection M′N′ with the normal are

$$x = \frac{x_1(a^2 - 2y_1^2)}{a^2 - b^2}, \quad y = \frac{y_1(2x_1^2 - b^2)}{a^2 - b^2}.$$

Squaring these expressions, and for x_1^2 putting $\frac{a^2}{b^2}(b^2 - y_1^2)$, we get two equations of the forms $y_1^6 + p_1y^4 \ldots - \frac{b^4(a^2 - b^2)^2y^2}{4a^4} = 0$,

$$y_1^6 + p_2y_1^4 + \ldots - \frac{b^2}{4a^2}\left[a^6 - (a^2 - b^2)^2x^2\right] = 0,$$

in which the absolute terms, being the products of all the roots, are equal; therefore, equating these two expressions and reducing, we find the required locus is the ellipse $a^2x^2 + b^2y^2 = a^8 / (a^2 - b^2)^2$.

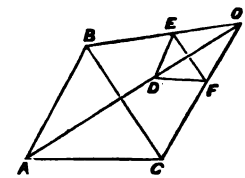
8323. (By MORGAN JENKINS, M.A. Suggested by Quest. 7756.)—Show how to construct a triangle of given form (sides proportional to a, b, c), having given three lines (h, k, l) severally equal to the distances from its angular points to a point in the plane of the triangle.

Solutions by (1) Dr. CURTIS; (2) *the* PROPOSER.

1. Construct the triangle ABC whose sides are a, b, c; find a point O, such that

$$OA : OB : OC = h : k : l,$$

viz., the intersection of the loci of vertices of two triangles whose bases and ratios of sides are given; take $OD = h$, $OE = k$, $OF = l$; then, evidently, the triangle DEF satisfies the required conditions.



2. The necessary and sufficient conditions for the possibility of solution are that the sum of any two of the rectangles ah, bk, cl, shall be greater than the third. This is a consequence of the six lines forming either an ordinary quadrilateral figure or a quadripunct, and is a slight extension of the theorem that in general the sum of the rectangles contained by the opposite sides of a quadrilateral figure is greater than the rectangle contained by the diagonals. (See TODHUNTER'S *Euclid*, Appendix 32, the construction of which article is here adapted in proof of the theorem involved. The further construction required for the problem has many analogies, in its use of similar triangles, with the construction used in the SYLVESTER-KEMPE extension of HART'S cell. See *Nature* for July 1 and July 15, 1875.)

Let ABC be a triangle having sides a, b, c; AO, BO, CO, lengths h, k, l, meeting in a point O, which in the figure is taken inside the triangle

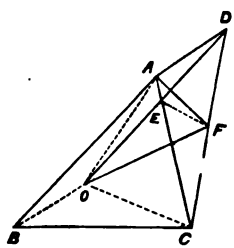


FIG. 1.

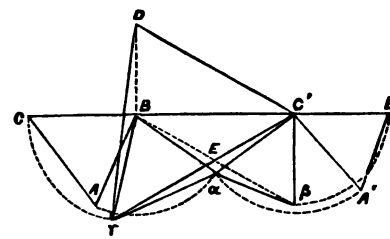


FIG. 2.

ABC. Draw AD equal and parallel to BO; let OD, equal and parallel to BA, cut AC in E; let the circles AED, OEC meet again in the point F: then the triangles AFD, CFO are similar triangles, for the angles AFD, CFO are equal, because they are equal to AED, CEO respectively; and the angle DAF is equal to DEF, which is the supplement of OEF and therefore equal to OCF. Hence the angle DFO is equal to the angle AFC; and since, from the similar triangles DFA, OFC, DF : FO = AF : FC, therefore also the triangles DFO, AFC are similar triangles, and DO : AC = DF : FA. Therefore AB : AC = DF : FA and angle BAC is equal to angle AED or AFD; therefore the triangle ABC is similar to each of the triangles DFA and OFC. Again, since the triangles FCO, ACB are similar, having equal angles at the common point C, and the other angles in the same order of rotation, the triangles AFC, BOC are also similar triangles. Hence AF : AC = BO : BC; and FO : OC =

AB : BC, that is, the sides of the triangle AOF are proportional to the rectangles AO, BC, BO, CA, and CO, AB; or $AO = h$, $AF = \frac{bk}{a}$, $FO = \frac{cl}{a}$, which gives the conditions stated.

Reversing the order of this construction.

Let CB, BC′, C′B′ (Fig. 2) be the three given lengths l, h, k in order, in the same straight line. Draw two triangles CAB, C′A′B′ of the given form with their sides respectively parallel. Let them swing round the points B, C′ respectively in opposite directions of rotation, until they fall in the positions $B\alpha\gamma$, $C'\alpha\beta$, having a common point at α. Then $B\alpha$, $C'\alpha$ are equal to $\frac{cl}{a}$ and $\frac{bk}{a}$ respectively, and the triangles $B\alpha\gamma$, $\beta\alpha C'$ are triangles of the required form, and $B\gamma$, $C'\beta$ are equal to l and κ respectively. Hence, if we draw BD equal and parallel to $\beta C'$, $D\gamma C'$ is the triangle required.

If $a = b = c$, the required triangle is to be equilateral, any two of the lengths h, k, and l must be greater than the third, and are the sides of the triangle $B\alpha C'$, the triangles $B\alpha\gamma$ and $\beta\alpha C'$ becoming equilateral triangles which may be either both external to the triangle $B\alpha C'$, or both described on the same side of $B\alpha$, $C'\alpha$ as the opposite vertices C′ and B respectively. The remainder of the construction is the same as before. It will be found, if H, K, L be the angles of the triangle formed of the lengths h, k, and l, and AO, BO, CO the same three lengths meeting at a point O in the plane of the required equilateral triangle ABC; then, if $H > \frac{2}{3}\pi$, $K < \frac{1}{3}\pi$, $L < \frac{1}{3}\pi$, the point O will be external to the triangle ABC, falling in a base compartment, whether the equilateral triangles are described on two sides of the triangle HKL, both externally or both internally.

If $H < \frac{2}{3}\pi$ but $> \frac{1}{3}\pi$, the construction with equilateral triangles applied externally to the triangle HKL will make the point O fall within the triangle ABC; but the construction, with equilateral triangles applied internally to the triangle HKL, will make the point O fall in a base compartment or a vertical compartment according as K, the intermediate angle, is $< \frac{1}{3}\pi$ or $> \frac{1}{3}\pi$. The compartments I refer to are those formed by the production of the sides of the triangle ABC,—a base compartment being one bounded by a base as BC, and by AB, AC produced; a vertical compartment the open angular space bounded by two sides as BA, CA produced backwards away from B and C.

I should like to notice an imaginary solution of this problem for the special case of the equilateral triangle. If $BOC = \theta$, $COA = \phi$, $AOB = \psi$, we have $k^2 - h^2 = 2l(k\cos\theta - h\cos\phi) = l(k\cos H - h\cos K)$, &c.,

whence $$\frac{\cos H - 2\cos\theta}{h} = \frac{\cos K - 2\cos\phi}{k} = \frac{\cos L - 2\cos\psi}{l},$$

or $$\frac{\cos H - 2\cos\theta}{\sin H} = \frac{\cos K - 2\cos\phi}{\sin K} = \frac{\cos L - 2\cos\psi}{\sin L}, \text{ say } = \mu.$$

Substituting in the cosine relation arising from the relation $\theta + \phi + \psi = 2\pi$, or $\theta = \phi + \psi$, &c., we have a cubic in μ, which ultimately reduces to

$$\mu = \pm\sqrt{3}, \text{ or } \mu = -(\cot H + \cot K + \cot L) = -\cot\chi,$$

where χ is the Brocard-angle.

The first two values of μ give the before-mentioned construction,

according as the equilateral triangles are applied externally or internally. The third value of μ gives no real construction, for then

$$\frac{\cos H - 2\cos\theta}{\sin H} = -\cot\chi = -\frac{1+\cos H\cos K\cos L}{\sin H\sin K\sin L},$$

and thence $$\cos\theta = 1 + \frac{(1-\cos H)[1-\cos(K\sim L)]}{2\sin K\sin L},$$

with similar values for $\cos\phi$ and $\cos\psi$; but these, being numerically greater than 1, furnish no real values of θ, ϕ, and ψ.

8389. (By Professor SYLVESTER, F.R.S.)—Prove that (1) if the roots of the equation $ax^4+4bx^3+6cx^2+4dx+e=0$ are spaced out at harmonic intervals [*i.e.*, if p, q, r, s being the roots, $q-p, r-p, s-p$ are in harmonic progression] the cubic invariant $\begin{vmatrix} abc \\ bcd \\ cde \end{vmatrix}$ will vanish; (2) if the roots of the equations $Ax^2+2Bx+C=0$, $A'x^2+2B'x+C'=0$, taken alternately, are spaced out at harmonic intervals, the invariant $AC'-2BB'+CA'$ will vanish; (3) the invariant $AC'-2BB'+CA'$ is contained as a divisor in the cubic invariant of the biquadratic function

$$(Ax^2+2Bx+C)(A'x^2+2B'x+C');$$

and form (4) the value of the remaining factor or quotient in (3).

Solution by R. F. DAVIS, M.A.; Professor MATHEWS, M.A.; *and others.*

1. If P, Q, R, S be four collinear points, and O the middle point of PR, then the necessary condition for (PQRS) being harmonic is

$$OQ.OS = OP^2 = OR^2.$$

The given equation will therefore have its roots spaced harmonically, if by diminishing them by the same quantity it can be transformed into the typical form $(x^2-\alpha^2)(x^2+4\beta x+\alpha^2)=0$, or $x^4+4\beta x^3+4\alpha^2\beta x-\alpha^4=0$. In this case the cubic invariant becomes $\begin{vmatrix} 1, & \beta, & 0 \\ \beta, & 0, & \alpha^2\beta \\ 0, & \alpha^2\beta, & \alpha^4 \end{vmatrix}$ which $\equiv 0$.

2. It is shown in SALMON'S *Conics* that the above condition may be written $2pr+2qs=(p+r)(q+s)$, where p, q, r, s are the distances of the four points from a fixed point on their line of collinearity. Hence, if

$$p+r=-\frac{2B}{A},\quad q+s=-\frac{2B'}{A'},\quad pr=\frac{C}{A},\quad qs=\frac{C'}{A'},$$

this relation can be written

$$\frac{2C}{A}+\frac{2C'}{A'}=\frac{4BB'}{AA'},\text{ or } AC'-2BB'+CA'=0.$$

3. The cubic invariant of the biquadratic function can, by putting $AC'+4BB'+CA'=6\lambda BB'$ (where λ is undetermined), be written in the form $\begin{vmatrix} AA', & \frac{1}{2}(AB'+BA'), & \lambda BB' \\ \frac{1}{2}(AB'+BA'), & \lambda BB', & \frac{1}{2}(BC'+CB') \\ \lambda BB', & \frac{1}{2}(BC'+CB'), & CC' \end{vmatrix}$,

or, when expanded and simplified,

$$(\lambda^3-3\lambda^2+2\lambda)\,B^3B'^3+(\lambda-1)\,BB'\{B^2\,.\,A'C'-AA'CC'+B'^2\,.\,AC\}.$$

Hence it vanishes when $\lambda = 1$, and consequently $AC'-2BB'+CA'$ is a factor.

4. The remaining factor (neglecting BB') is

$$(\lambda^2-2\lambda)\,B^2B'^2+B^2A'C'-AA'CC'+B'^2AC,$$

or, restoring λ its value,

$$(AC'+4BB'+CA')^2-12BB'(AC'+4BB'+CA')$$
$$+36\{B^2\,.\,A'C'-AA'CC'+B'^2\,.\,AC\},$$

or
$$(AC'-2BB'+CA')^2-36\,(B^2-AC)\,(B'^2-A'C').$$

This factor could also be *à priori* obtained as follows.

The factor $AC'-2BB'+CA'$ vanishes when $2pr+2qs=(p+r)(q+s)$, or when $3\,(pr+qs) = M$ = sum of products of the roots taken two at a time $= \dfrac{AC'+4BB'+CA'}{AA'}$. As there can be no distinction in dealing with the general equation between its four roots, the other factor must arise from the product $\{M-3\,(pq+rs)\}\,\{M-3\,(ps+qr)\}$,

or
$$M^2-3M\,(p+r)\,(q+s)+9\{pr\,(q^2+s^2)+qs\,(p^2+r^2)\}.$$

By substituting for p, q, r, s, this result will be found to agree with the one previously obtained.

8390. (By Professor CROFTON, F.R.S.) — If m things are divided among n persons so that every one receives something, prove that (1) the number of ways this can be done is equal to the function $\Delta^n 0^m$ in the Calculus of finite differences, viz.,

$$\Delta^n 0^m = n^m-n\,(n-1)^m+\tfrac{1}{2}n\,(n-1)(n-2)^m-\&c.,$$

which vanishes when $m<n$. Find (2) the number of ways, if a particular set of r out of the n persons are all to receive something; also (3) if simply a number r, at least, are to do so; again (4), suppose all the things are to go to a number r, each one receiving something.

Solution by FREDERIC R. J. HERVEY.

(1) Let the number of ways in which m things can be divided among r persons, each one receiving something, be denoted by m $(=0$ if $m<r)$. Now, the whole number of ways in which m things can be divided among n persons is made up of those in which all go to one, those in which only two have anything, those in which only three have, and so on. The number of ways in which all the things go to a particular set of r persons is m_r; of those in which all go to r persons (not specified) the number is $\dfrac{n!}{r!\,(n-r)!}m_r$. Therefore

$$n^m = nm_1+\tfrac{1}{2}n\,(n-1)\,m_2+\ldots+\frac{n\,(n-1)\ldots(n-m+1)}{m!}m_m,$$

which, for any one value of m, must be identical, term by term, with

$$n^m = n\Delta 0^m + \tfrac{1}{2}n(n-1)\,\Delta^2 0^m + \ldots + \frac{n(n-1)\ldots(n-m+1)}{m!}\,\Delta^m 0^m,$$

(as may be seen by making n successively $= 1, 2, 3, \ldots$), a well-known expansion. Hence $m = \Delta^r 0^m$ for all values of m and r.

(2) All may go to the r specified persons, giving $\Delta^r 0^m$ ways; or any one of the remaining $n-r$ persons may receive something, giving $(n-r)\,\Delta^{r+1}0^m$ ways; or any two others, giving $\frac{1}{2}(n-r)(n-r-1)\,\Delta^{r+2}0^m$ ways; and so on. Hence the whole number of ways is

$$\Delta^r 0^m + (n-r)\,\Delta^{r+1}0^m + \tfrac{1}{2}(n-r)(n-r-1)\,\Delta^{r+2}0^m + \ldots + \Delta^n 0^m.$$

(3, 4). If simply a number r (and no more) are each to receive something, the number of ways is $\dfrac{n!}{r!\,(n-r)!}\,\Delta^r 0^m$; if r, at least, are to do so, the number is

$$\frac{n!}{r!\,(n-r)!}\,\Delta^r 0^m + \frac{n!}{(r+1)!\,(n-r-1)!}\,\Delta^{r+1}0^m + \ldots + \Delta^n 0^m.$$

8393. (By Professor Neuberg.)—Soient I, E, E′, E″ le centre du cercle inscrit et ceux des cercles ex-inscrits au triangle ABC, et soient F, F′, F″ les points où les bissectrices des angles BIC, CIA, AIB, rencontrent respectivement BC, CA, AB. Démontrer que les droites EF, E′F′, E″F″ se rencontrent en un même point L tel que les parallèles menées par L aux côtés du triangle EE′E″ et limitées respectivement par les angles du triangle ABC sont égales entre elles.

Solution by R. F. Davis, M.A.; W. J. Barton, M.A.; *and others.*

Employing trilinear coordinates; at F, $\beta : \gamma = \cos\frac{1}{2}C : \cos\frac{1}{2}B$, for $BF : CF = BI : CI = \operatorname{cosec}\frac{1}{2}B : \operatorname{cosec}\frac{1}{2}C$. The coordinates of E are $(-1, 1, 1)$, and the equation to EF

$$\alpha(\cos\tfrac{1}{2}B - \cos\tfrac{1}{2}C) + \beta\cos\tfrac{1}{2}B - \gamma\cos\tfrac{1}{2}C = 0.$$

Similarly the equations to E′F′, E″F″. The coordinates of the point L through which each of these three straight lines passes are

$$\alpha' = \cos\tfrac{1}{2}B + \cos\tfrac{1}{2}C - \cos\tfrac{1}{2}A, \quad \beta' = \&c., \quad \gamma' = \&c.$$

Whence $$(\beta' + \gamma')\sec\tfrac{1}{2}A = \ldots = \ldots, \ \&c.$$

8398. (By Professor Mathews, M.A.)—If i, w, v are defined by the equations $i^2 + 1 = 0$, $w^2 + w + 1 = 0$, $v = i + w$, prove that

$$w = -\frac{v^4 + 4v^3 + 8v^2 + 12v + 3}{4v^3 + 6v^2 + 16v + 7}, \quad i = \frac{5v^4 + 10v^3 + 24v^2 + 19v + 3}{4v^3 + 6v^2 + 16v + 7}.$$

Solution by Prof. TANNER, M.A.; W. J. BARTON, M.A.; *and others.*

We have $(v-w)^2 = i^2$, $1+w^2-2vw+v^2 = 0$, $-(2v+1)w+v^2 = 0$;
whence $w = v^2/2v+1$; $i, = v-w, = v(v+1)/2v+1$;
and there is an equation for v, $0 = v^4+2v^3+5v^2+4v+1 = V$, say. The forms given by Professor MATHEWS are equivalent to these; viz., they are $w = v^2(2v^2+2v+7)-3V(+)$, $i = v(v+1)(2v^2+2v+7)+3V(+)$, where the denominator is $(2v+1)(2v^2+2v+7)$.

8407. (By Captain H. BROCARD.)—Les lignes qui joignent les sommets du triangle ABC aux points de Brocard ω et ω' déterminent sur les côtés opposés deux systèmes de trois points, qui sont les sommets de deux triangles, inégaux et dissemblables, mais équivalents. Trouver le rapport des surfaces de ces triangles à celle du triangle donné.

Solution by Professor DE LONGCHAMPS, *and* W. J. BARTON, M.A.

Considérons l'un des points de Brocard ω et posons

$$\text{aire BOC} = \alpha, \quad \text{aire COA} = \beta, \quad \text{aire AOB} = \gamma;$$

α, β, γ représentant par conséquent les coordonnées barycentriques de ω. Nous savons que $b^2\alpha = c^2\beta = a^2\gamma$, a, b, c désignant, bien entendu, les longueurs des côtés de ABC.

Cela posé, appelons A', B', C' les points de rencontre des droites ωA, ωB, ωC respectivement avec les côtés BC, CA, AB et proposons-nous d'évaluer l'aire de A'B'C'.

Nous avons $\frac{A'B}{A'C} = \frac{\gamma}{\beta}$, et par suite $A'B = \frac{a\gamma}{\beta+\gamma}$, $A'C = \frac{a\beta}{\beta+\gamma}$. Nous trouvons de même

$$B'C = \frac{ba}{\gamma+\alpha}, \quad C'A = \frac{b\gamma}{\gamma+\alpha}, \quad C'A = \frac{c\beta}{\alpha+\beta}, \quad A'B = \frac{c\alpha}{\alpha+\beta}.$$

De ces formules nous déduisons

$$\text{aire A'CB'} = \tfrac{1}{2}CA' \cdot CB' \sin C = \tfrac{1}{2}ab \sin C \, \frac{a\beta}{(\alpha+\gamma)(\beta+\gamma)}.$$

Soit S l'aire de ABC, nous avons donc $\frac{\text{aire A'CB'}}{S} = \frac{a\beta}{(\alpha+\gamma)(\beta+\gamma)}$, et, de même, $\text{aire}\frac{B'AC'}{S} = \frac{\beta\gamma}{(\beta+\alpha)(\gamma+\alpha)}$, $\text{aire}\frac{C'BA'}{S} = \frac{\gamma\alpha}{(\gamma+\beta)(\alpha+\beta)}$.

Ces égalités donnent, par combinaison,

$$\frac{S-A'B'C'}{S} = \frac{a\beta(\alpha+\beta)+\beta\gamma(\gamma+\beta)+a\gamma(\alpha+\gamma)}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)},$$

ou encore

$$\frac{A'B'C'}{S} = \frac{2a\beta\gamma}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)} \quad \ldots\ldots\ldots\

Le point ω' est déterminé par les relations $c^2\alpha' = a^2\beta' = b^2\gamma'$; la formule (A) donne pour le triangle A″B″C″, qui correspond à ω', la même expression que celle déjà trouvée pour A′B C′, d'où l'on conclut, finalement,

$$\frac{A''B''C''}{S} = \frac{A'B'C'}{S} = \frac{2a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}.$$

On peut observer que le résultat est encore le même, en appliquant la formule (A) au *point de Lemoine*.

Remarque.—En appelant *triangle associé* d'un point donné ω, dans le plan d'un triangle fondamental ABC, le triangle qui a pour sommets les points de rencontre des droites ωA, ωB, ωC avec les côtés de ce triangle fondamental, la proposition précédente s'énoncerait sous la forme brève et frappante : ***Les triangles associés aux points de Brocard et au point de Lemoine sont équivalents.***

J'ai dit tout à l'heure que la formule (A) permettait de voir des conséquences diverses. En voici une que j'ai indiquée autrefois (*Journal de Mathématiques élémentaires*, 1887, p. 224.)

Observons d'abord que la formule (A) peut être généralisée de la manière suivante. Prenons sur les côtés du triangle ABC trois points quelconques A′, B′, C′ ; si nous posons

$$\frac{A'B}{A'C} = -\alpha_1, \quad \frac{B'C}{B'A} = -\beta_1, \quad \frac{C'A}{C'B} = -\gamma_1,$$

le calcul que nous avons fait plus haut donne, avec cette notation nouvelle,

$$\frac{A'B'C'}{S} = \frac{-\alpha_1\beta_1\gamma_1 + 1}{(-\alpha_1+1)(-\beta_1+1)(-\gamma_1+1)} \quad \text{.....(B).}$$

Le second membre de cette formule conserve la même valeur quand on change α_1 en $1/\alpha_1$, et de même β_1 et γ_1 en $1/\beta_1$ et $1/\gamma_1$. D'où cette proposition : ***Deux triangles A′B′C′, A″B″C″ inscrits à un triangle ABC de telle sorte que les milieux des côtés de ce triangle soient aussi les milieux des droites A′A″, B′B″, C′C″, sont équivalents.***

Lorsque les droites AA′, BB′, CC′ concourent en un certain point O ; les droites AA″, BB″, CC″ concourent aussi en un point O′ ; ces points O, O′, ainsi associés, sont ceux que nous avons nommés *points réciproques*. La proposition précédente donne donc le corollaire suivant :

Les triangles associés à deux points réciproques sont équivalents.

Nous ferons enfin observer que le théorème de Menelaus doit être considéré comme constituant un cas particulier du théorème général qui correspond à cette formule. En effet, comme je l'ai fait remarquer dans mon *Essai sur la Géométrie de la règle et de l'équerre* (§ 8), si A′B′C′ a une surface nulle, on a $\alpha_1\beta_1\gamma_1 = 1$; et *réciproquement*, si $\alpha_1\beta_1\gamma_1 = 1$, on a A′B′C′ = 0 ; les trois points A′, B′, C′ sont en ligne droite.

[If A_1, B_1, C_1 be the system formed by producing Aω, Bω, Cω to meet the opposite sides, we have $A_1C = \frac{ab^2}{a^2+b^2}$; also, if α', γ' are the coordinates of B_1, $\frac{\alpha'}{b^2} = \frac{\gamma'}{ac} = \frac{2\Delta}{a(b^2+c^2)}$; hence the areas of A_1B_1C, AB_1C_1, BA_1C_1, are $\frac{1}{2}\alpha' \cdot A_1C = \frac{\Delta b^4}{(a^2+b^2)(b^2+c^2)}$, $\frac{\Delta c^4}{(b^2+c^2)(c^2+a^2)}$, $\frac{\Delta a^4}{(c^2+a^2)(a^2+b^2)}$;

therefore $\frac{A_1B_1C_1}{\Delta} = 1 - \Sigma\frac{a^4}{(c^2+a^2)(a^2+b^2)} = \frac{2a^2b^2c^2}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$.]

7896. (By the Editor.)—In a circle of radius a, a chord QRQ′ moves perpendicularly to a chord AX, whose central distance (CS) is $a \sin \alpha$, and from A tangents AP, AP′ are drawn to circles on QR, Q′R as diameters; show that (1) the locus of the points P, P′ is a heart-shaped curve whose polar equation, with A as pole and AX as polar axis, is

$$r(1 + 4\tan^2 \tfrac{1}{2}\theta) = 2a(\cos\alpha + 2\sin\alpha \tan \tfrac{1}{2}\theta);$$

(2) when AX is a diameter, this cardioid is $r(1 + 4\tan \frac{1}{2}\theta) = 2a$, which cuts the circle at a point T (whose distance from A is $AT = \frac{8}{9}a$, and whose abscissa is $AD = \frac{2}{3}a$), has a maximum ordinate VE at a point V whose abscissa is $AE = (\cdot 934)\,a$, extends to a distance $AZ = (\cdot 078)\,a$ to the left of A, and contains an area equal to eight-ninths of the area of the circle; (3) the equations of the cardioid between (x, r) and (y, r) are

$$\frac{x}{r} = \frac{5r - 2a}{3r + 2a}, \quad \frac{y}{r} = \frac{4r^{\frac{1}{2}}(2a - r)^{\frac{1}{2}}}{3r + 2a};$$

(4) if PG be a perpendicular to QR, then $(RG : GQ) = 4\,(AR : RX)$; (5) the locus of the intersection K of the tangents from A, X to the variable circle is an ellipse of which A, X are the foci and the semi-axes $\frac{4}{3}a$, $\frac{4}{9}a$; (6) the distance from K to either of the points of contact of these tangents is two-thirds of the radius of the given circle.

Solution by the Proposer.

1. Let $AP = AR = r$, $\angle PAX = \theta$, then $OR = \frac{1}{2}QR = r\tan\frac{1}{2}\theta$,
$AR \cdot RX = QR \cdot RQ' = QB^2 - BR^2$,
therefore $r(2a\cos\alpha - r)$
$= (2r\tan\frac{1}{2}\theta - a\sin\alpha)^2 - (a\sin\alpha)^2$
$= 4r\tan\frac{1}{2}\theta\,(r\tan\frac{1}{2}\theta - a\sin\alpha)$,
whence follows the stated equation of the locus. The curve is of the form XPTYAT′P′X′ with a cusp at A.

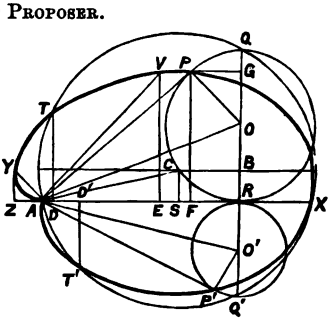


Fig. 1.

2. When AX is a diameter we have $\alpha = 0$, the equation becomes that given, and the area of the curve is

$$2\int_0^\pi \tfrac{1}{2}r^2 d\theta = 4a^2 \int_0^\pi \frac{d\theta}{(1 + 4\tan^2\frac{1}{2}\theta)^2}$$
$$= [\text{putting } z = \tan\tfrac{1}{2}\theta]\; 8a^2 \int_0^\infty \frac{dz}{(1 + z^2)(1 + 4z^2)^2}$$
$$= \left\{ \frac{16a^2 z}{3(1 + 4z^2)} + \tfrac{8}{3}a^2(\tan^{-1}2z + \tan^{-1}z) \right\}_{z=0}^{z=\infty} = \tfrac{8}{9}\pi a^2$$
$$= \tfrac{8}{9}\,(\text{area of circle AQXQ}').$$

For the angular ordinate ($\theta_1 = TAX$) of the intersection of the cardioid with the circle, we have $\cos\theta_1(1 + 4\tan^2\frac{1}{2}\theta_1) = 1$,

$$4\tan^2\tfrac{1}{2}\theta_1 = \frac{1 - \cos\theta_1}{\cos\theta_1} = \frac{2\sin^2\frac{1}{2}\theta_1}{\cos\theta_1}, \quad 2\cos\theta_1 = \cos^2\tfrac{1}{2}\theta_1 = \tfrac{1}{2}(1 + \cos\theta_1),$$

whence $\cos\theta_1 = \frac{1}{3}$, $AT = 2a\cos\theta_1 = \frac{2}{3}a$, $AD = AT\cos\theta_1 = \frac{2}{9}a$.

The ordinate will be the greatest when

$$y = r\sin\theta = \frac{2a\sin\theta}{1+4\tan^2\frac{1}{2}\theta} \text{ is a maximum,}$$

that is, putting $v = \log\left(\frac{y}{2a}\right) = \log\sin\theta - \log(1+4\tan^2\frac{1}{2}\theta)$,

when $\frac{dv}{d\theta} \equiv \cot\theta - \frac{4\tan\frac{1}{2}\theta\sec^2\frac{1}{2}\theta}{1+4\tan^2\frac{1}{2}\theta} = 0$, or if $z = \tan\frac{1}{2}\theta_2 = \tan\frac{1}{2}\text{VAX}$,

when $\frac{1-z^2}{2z} = \frac{4z^2(1+z^2)}{1+4z^2}$, or $12z^4+5z^2=1$,

therefore $z^2 = \frac{1}{24}(\sqrt{73}-5)$, $\text{AV} = \frac{2a}{1+4z^2} = \frac{1}{8}(\sqrt{73}-1)a = (1{\cdot}257)\,a$,

$\cos\theta_2 = \frac{1-z^2}{1+z^2} = \frac{1}{8}(13-\sqrt{73})$, $\text{AE} = \frac{1}{16}(7\sqrt{73}-43)a = (\cdot 934)\,a$.

The distance (AZ) to the left of A where the tangent AY cuts the axis will be obtained by minimising $x = r\cos\theta = \frac{2a\cos\theta}{1+4\tan^2\frac{1}{2}\theta}$, that is, by minimising $v \equiv \log\left(\frac{x}{2a}\right) = \log\cos\theta - \log(1+4\tan^2\frac{1}{2}\theta)$,

therefore $-\frac{dv}{d\theta} \equiv \tan\theta + \frac{4\tan\frac{1}{2}\theta\sec^2\frac{1}{2}\theta}{1+4\tan^2\frac{1}{2}\theta} \equiv \frac{2z}{1+z^2} + \frac{4z(1+z^2)}{1+4z^2} = 0$,

where $z = \tan\frac{1}{2}\theta$; hence $2z^4-4z^2=3$, and $z^2 = 1+\frac{1}{2}\sqrt{10}$; thus, if $\angle\,\text{YAZ} = \theta_3$, we have $\cos\theta_3 = \frac{z^2-1}{z^2+1} = \frac{1}{3}(2\sqrt{10}-5)$,

and $\text{AZ} = \frac{1}{3}\text{AY}(2\sqrt{10}-5) = \frac{8}{15}a(13-4\sqrt{10}) = (\cdot 078)\,a$.

3. Putting $\angle\,\text{QAX} = \phi$, we have

$$\tan\tfrac{1}{2}\theta = \tfrac{1}{2}\tan\phi, \quad \cos\theta = \frac{4-\tan^2\phi}{4+\tan^2\phi}, \quad \sin\theta = \frac{4\tan\phi}{4+\tan^2\phi};$$

also $\tan^2\phi = \frac{\text{QR}^2}{\text{AR}^2} = \frac{\text{AR}\,.\,\text{RX}}{\text{AR}^2} = \frac{\text{RX}}{\text{AR}} = \frac{2a-r}{r}$,

therefore $\frac{x}{r} = \frac{5r-2a}{3r+2a}$, $\frac{y}{r} = \frac{4r^{\frac{1}{2}}(2a-r)^{\frac{1}{2}}}{3r+2a}$ .....(α, β).

By minimising x and maximizing y, in the usual way, from these equations, we find that y has its maximum value VE when $r = \frac{1}{8}(\sqrt{73}-1)a$, and that x has its minimum value

$$\text{AZ} = \tfrac{8}{15}(13-4\sqrt{10})\,a \text{ when } r = \tfrac{8}{15}(2\sqrt{10}-5)\,a,$$

which agree with the results otherwise obtained in (2).

4. From equation (β) we have

$$\frac{4r}{3r+2a} = \frac{y}{(2ar-r^2)^{\frac{1}{2}}} = \frac{\text{RG}}{\text{RQ}}, \quad \text{whence} \quad \frac{\text{RG}}{\text{GQ}} = \frac{4r}{2a-r} = 4\frac{\text{AR}}{\text{RX}}.$$

5. Let the tangents AP, XH to the circle (O) meet in K (Fig. 2); then, putting

$$\angle OAX = \chi, \quad \angle OXA = \psi,$$

and (x, y) for the coordinates of K measured along MX, ML (M being the mid-point of AX), we have

$$ON \cot\chi \,.\, ON \cos\psi = AN \,.\, NX = QN^2 = 4ON^2;$$

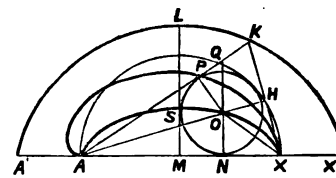


FIG. 2.

therefore
$$\cot\chi \cot\psi = 4 \quad \ldots\ldots(1),$$

also
$$\frac{2(a+x)}{y} = 2\cot 2\chi = \cot\chi - \tan\chi \quad \ldots\ldots(2),$$

and
$$\frac{2(a-x)}{y} = \cot\psi - \tan\psi = 4\tan\chi - \tfrac{1}{4}\cot\chi \quad \ldots\ldots(3);$$

therefore
$$15\tan\chi = \frac{2}{y}(5a-3x), \quad \frac{15}{4}\cot\chi = \frac{2}{y}(5a+3x);$$

hence the equation of the locus of K is

$$\frac{225}{4} = \frac{4}{y^2}(25a^2 - 9x^2) \quad \text{or} \quad \frac{x^2}{25} + \frac{y^2}{16} = \frac{a^2}{9},$$

which designates an ellipse A'LKX' with MA' $(= \frac{5}{3}a)$ and ML $(= \frac{4}{3}a)$ for lengths of the semi-axes, and focal distance from centre

$$= (\tfrac{25}{9}a^2 - \tfrac{16}{9}a^2)^{\frac{1}{2}} = a = MA = MX.$$

6. In Fig. 2 we have $KA + KX = \frac{10}{3}a$, and $AP + XH = AX = 2a$, hence $KP = KX = \frac{4}{3}a$.

The locus of O is, of course, an ellipse ASOX, the lengths of whose semi-axes are $MX = a$, $MS = \frac{1}{3}a$.

8387. (By H. MACCOLL, B.A.)—Let u and v be often taken at random within certain limits, and for each pair of these random values let the equations $f(x, y) = u$, and $\phi(x, y) = v$ determine the position of a point (or points) referred to Cartesian axes. These points will ultimately form some shaded geometrical figure, and the shading, depending on the functions f and ϕ, will not, generally speaking, be uniform. Find an expression for the relative depth of shading round any specified point x, y.

Solution by the Rev. H. C. POCKLINGTON, M.A.

We will suppose u, v coordinates of points in a plane (the axes of reference being parallel to those in the plane xy). The density in this plane will be uniform.

The density at a point x, y is obviously the reciprocal of the ratio of a small area about x, y to the corresponding area in the plane u, v.

Let us take as our first area the triangle formed by

$$(x, y), \quad (x + dx, y), \quad (x, y + dy);$$

its area is
$$\pm \tfrac{1}{2}\sin\omega \, dx \, dy \quad \ldots\ldots (1).$$

When x increases by dx (y being constant), u increases by $\frac{df}{dx}dx$, and v by $\frac{d\phi}{dx}dx$. Similarly, if y increases by dy (x remaining constant), u increases by $\frac{df}{dy}dy$, and v by $\frac{d\phi}{dy}dy$; therefore the area of the triangle in the plane uv is $\pm\frac{1}{2}\sin\omega\left(\frac{df}{dx}dx\,.\,\frac{d\phi}{dy}dy-\frac{d\phi}{dx}dx\,.\,\frac{df}{dy}dy\right)$

$$=\pm\tfrac{1}{2}\sin\omega\,dx\,dy\left(\frac{df}{dx}\frac{d\phi}{dy}-\frac{df}{dy}\frac{d\phi}{dx}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

The density $D = (2) \div (1) = \pm\left(\frac{df}{dx}\frac{d\phi}{dy}-\frac{d\phi}{dx}\frac{df}{dy}\right) = \pm\begin{vmatrix}\frac{df}{dx} & \frac{d\phi}{dx}\\ \frac{df}{dy} & \frac{d\phi}{dy}\end{vmatrix}$

where the sign is so taken as to make D positive.

8417. (By W. H. Blythe, M.A.)—Find whether the equation $\frac{d^2w}{dx^2}+\frac{2}{x}\frac{dw}{dx}+\left(\frac{2}{x^2}-n^2\right)w=0$ has any real solution.

Solutions by (1) R. Rawson, F.R.A.S.; (2) A. R. Forsyth, M.A.

1. It will be necessary to solve a more general differential equation than the one given in the question, by the help of the method of *Integration* by *Predication*; that is, if two equations (A) and (C) are cognate by the known relation (B) connecting them, then (B), the middle term, is the *Predicate* of (C).

When, therefore, a solution of either (A) or (C) is given, the other is *predicated* by means of (B); *ex. gr.*,

$$P\frac{du}{dx}+\left\{PQ+PZ+v\right\}w=y,\quad \frac{dy}{dx}+\left\{Q-\frac{dP}{Pdx}-Z-\frac{v}{P}\right\}y=0\ldots(A,B),$$

where P, Q, Z, v are functions of x.

Eliminate (y) from (A) and (B) by differentiation, then we have

$$\frac{d^2u}{dx^2}+2Q\frac{du}{dx}+\left\{\frac{dQ}{dx}+Q^2+\frac{dZ}{dx}-Z^2+\frac{dv}{Pdx}-\frac{v}{P}\frac{dP}{Pdx}-\frac{2vZ}{P}-\frac{v^2}{P^2}\right\}w=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(C).$$

Equation (C) is the cognate of (A), the solution of which is also the solution of (C) by means of the *predicate* (B).

Of course (A) and the *predicate* (B) can be infinitely varied. Integrate (A) and (B) in the usual way—

$$y=cPe^{\int\left(Z+\frac{v}{P}\right)dx},\qquad e^{\int\left(Z+\frac{v}{P}+Q\right)dx}w=\int e^{\int\left(Z+\frac{v}{P}+Q\right)dx}\frac{y}{P}dx+c_1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots(1,\ 2).$$

Let $Q = \frac{p}{x}$, $Z = -\frac{i}{x}$, therefore $\frac{dQ}{dx} = -\frac{p}{x^2}$ and $\frac{dZ}{dx} = \frac{i}{x^2}$.

Put $$\frac{dP}{dx} - \frac{2i}{x}P + \frac{h}{v}P^2 = -v \quad \text{.....(3)};$$

and substitute these values in (C), then

$$\frac{d^2w}{dx^2} + \frac{2p}{x}\frac{dw}{dx} + \left\{\frac{p(p-1)+i(i+1)}{x^2} + h\right\}w = 0 \quad \text{.....(4)},$$

where p, i, h are constants. It now remains to solve (3), which is readily done as follows. If a, b, c are constants, let

$$\frac{dU}{dx} + \frac{aU}{x} + bx^n U^2 = cx^m, \quad bU = \frac{cx^{m-n}}{y} + \frac{n+1-a}{x^{n+1}} \quad \text{.....(A}_1\text{ B}_1\text{)}.$$

From (A_1), (B_1) there results the cognate

$$\frac{dy}{dx} + \frac{a_1}{x}y + bx^n y^2 = cx^m \quad \text{.....(C}_1\text{)},$$

where $a_1 = a-(n+m+2)$. If, therefore, i of these transformations be made, there results

$$\frac{dy_{i-1}}{dx} + \frac{a-i(n+m+2)}{x}y_{i-1} + bx^n y_{i-1}^2 = cx^m \quad \text{.....(C}_{i-1}\text{)}.$$

Now, if (A_1) is soluble, so is (C_{i-1}). In (A_1) put $a = 0$, $n = 0$, $m = 0$, then (C_{i-1}) becomes

$$\frac{dy_{i-1}}{dx} - \frac{2i}{x}y_{i-1} + by_{i-1}^2 = c \quad \text{.....(5)},$$

which is soluble, because (C_1) is soluble. Equation (5), therefore, coincides with (3), when $bv = h$, $c = -v$.

The conclusion is, there is a real solution to (4); that is, equation (4) is soluble by continued transformations of (A_1), (B_1), (C_1), as above indicated. Put $p = 1$, $i = 1$ in (4).

Hence $$\frac{d^2u}{dx^2} + \frac{2}{x}\frac{du}{dx} + \left\{\frac{2}{x^2} + h\right\}w = 0 \quad \text{.....(6)}$$

has a real solution—of course (h) is any constant positive or negative.

From a slight alteration in (A) and (B), viz.,

$$\frac{du}{dx} + \left(Q - \frac{dv}{vdx}\right)w = y, \quad \frac{dy}{dx} + \left(Q + \frac{dv}{vdx}\right)y = 0 \quad \text{.....(A}_2\text{, B}_2\text{)},$$

it follows that $$\frac{d^2u}{dx^2} + 2Q\frac{du}{dx} + \left\{\frac{dQ}{dx} + Q^2 - \frac{d^2v}{vdx^2}\right\}w = 0 \quad \text{.....(C}_2\text{)}.$$

From (A_2) and (B_2), by integration,

$$vy = ce^{-\int Q\,dx}, \quad e^{\int Q\,dx}w = cv\int\frac{dx}{v^2} + c_2 v \quad \text{..... (7, 8)}.$$

Put $Qx = -i$, and $$\frac{d^2v}{v\,dx^2} = \frac{i(i+1)}{x^2} \mp h^2 \quad \text{.....(9)}.$$

Then (C_2) becomes $$\frac{d^2u}{dx^2} - \frac{2i}{x}\frac{du}{dx} \pm h^2 w = 0 \quad \text{.....(10)}.$$

Put $Qx = i + 1$, then (C_2) becomes

$$\frac{d^2u}{dx^2} + \frac{2(i+1)}{x}\frac{du}{dx} \pm h^2 w = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11).$$

Equations (10) and (11) were integrated by Professor WILLIAMSON by the symbolic method δ.

Equation (9) is known as GASKIN'S equation, of which many solutions have been given (see Mr. GLAISHER'S on RICCATI'S Equation and Transformations, p. 812 from the *Philosophical Transactions of the Royal Society*, Part III., 1881).

The following are integrated by the above method—

$$\frac{d^2u}{dx^2} + \frac{2i}{x}\frac{du}{dx} \pm h^2 u = 0, \quad \frac{d^2u}{dx^2} - \frac{2(i+1)}{x}\frac{du}{dx} \pm h^2 w = 0 \ldots\ldots (12, 13).$$

2. *Otherwise* :—The equation is probably a misquotation of Ex. 1 (i.), p. 125, of my *Differential Equations*, which is

$$\frac{d^2\omega}{dx^2} + \frac{2}{x}\frac{d\omega}{dx} - \left(\frac{2}{x^2} - n^2\right)\omega = 0.$$

This was included as an example of the method of variation of parameters, and the solution is by that method obtainable in the form

$$\omega = \frac{d}{dx}\left\{\frac{E\cos nx + F\sin nx}{x}\right\}.$$

A shorter method of solving it, however, is the following. Reducing the equation to its normal form by the substitution

$$z = \omega^{\frac{1}{2}\int \frac{2}{x}dx} = \omega x,$$

we find the equation in z to be

$$\frac{d^2z}{dx^2} + \left(n^2 - \frac{2}{x^2}\right)z = 0,$$

the solution of which is (p. 175 of my book)

$$z = x\frac{d}{dx}\left\{\frac{E\cos nx + F\sin nx}{x}\right\},$$

and leads to the above given value for ω.

Any linear differential equation of the given form is integrable in series, and therefore has a real solution. If in deducing the solution by any method imaginary quantities should enter, it is desirable that the form should be modified by the modification of arbitrary constants (as, for instance, in § 47).

The form of the series for the equation

$$\frac{d^2\omega}{dx^2} + \frac{2}{x}\frac{d\omega}{dx} + \left(\frac{2}{x^2} - n^2\right)\omega = 0$$

is evidently $\omega = x^\mu\left\{A_0 - A_1x^2 + A_2x^4 - \ldots + (-1)^r A_r x^{2r} + \ldots\right\}$,

where μ is determined by $\mu(\mu-1) + 2\mu + 2 = 0$, and the relation between the successive coefficients is

$$A_r = \frac{n^2}{(\mu+2r)(\mu+2r+1)+2}A_{r-1} = \frac{n^2}{2r(2r+2\mu+1)}A_{r-1}.$$

The roots of the equation which determines μ are

$$\alpha = -\tfrac{1}{2} + \tfrac{1}{2}i\sqrt{7}, \quad \beta = -\tfrac{1}{2} - \tfrac{1}{2}i\sqrt{7},$$

so that two special solutions of the equation are given by

$$x^{\alpha}\left\{1-\frac{n^2x^2}{2\,(2+i\sqrt{7})}+\frac{n^4x^4}{2\,.\,4\,(2+i\sqrt{7})\,(4+i\sqrt{7})}-\ldots\right\},$$

$$x^{\beta}\left\{1-\frac{n^2x^2}{2\,(2-i\sqrt{7})}+\frac{n^4x^4}{2\,.\,4\,(2-i\sqrt{7})(4-i\sqrt{7})}-\ldots\right\}.$$

Now $x^{\alpha}=x^{-\frac{1}{2}}\left\{\cos\left(\frac{1}{2}\sqrt{7}\log x\right)+i\sin\left(\frac{1}{2}\sqrt{7}\log x\right)\right\}$,

and x^{β} is this quantity with the sign of i changed. Let the expression with the brackets in the former solution be denoted by $\phi(x^2)+i\psi(x^2)$; that in the latter is $\phi(x^2)-i\psi\ x^2)$.

Hence our special solutions become

$$x^{-\frac{1}{2}}\left\{\phi(x^2)\cos\left(\tfrac{1}{2}\sqrt{7}\log x\right)-\psi(x^2)\sin\left(\tfrac{1}{2}\sqrt{7}\log x\right)\right\}$$

$$x^{-\frac{1}{2}}\left\{\phi(x^2)\sin\left(\tfrac{1}{2}\sqrt{7}\log x\right)+\psi(x^2)\cos\left(\tfrac{1}{2}\sqrt{7}\log x\right)\right\};$$

and the primitive therefore is

$$\omega x^{\frac{1}{2}}=\mathrm{C}\left\{\phi(x^2)\sin\left(\tfrac{1}{2}\sqrt{7}\log x+\alpha\right)+\psi(x^2)\cos\left(\tfrac{1}{2}\sqrt{7}\log x+\alpha\right)\right\},$$

where ϕ and ψ are given by

$$\phi(x^2)+i\psi(x^2)=1-\frac{n^2x^2}{2\,(2+i\sqrt{7})}+\frac{n^4x^4}{2\,.\,4\,(2+i\sqrt{7})(4+i\sqrt{7})}-\ldots.$$

These expressions are connected with Bessel's Functions of imaginary order; but it is unnecessary to enter into their particular forms, as the example asked by Mr. Blythe is contained as a very special case of Ex. 15, p. 179, the solution of which, due to Lommel, is there given.

[Mr. Rawson remarks that, "After what has been said by one of the greatest authorities alive, Professor Sylvester, in commendation of Mr. Forsyth's *Differential Equations*, from which work the question is taken, ('In my opinion, the best written mathematical book extant in the English language'), there remains little more to be said in its praise. The question is not probably a misquotation of Ex. 1 (i.), p. 125, of Mr. Forsyth's *Differential Equations*, but is probably taken from Ex. (β), p. 69. The solution of Ex. 1 (i.), p. 125, by the method of Predication, is to assume $z=\frac{i}{x}$, when (4) becomes

$$\frac{d^2\omega}{dx^2}+\frac{2p}{x}\frac{d\omega}{dx}+\left\{\frac{p\,(p-1)-i\,(i+1)}{x^2}+h\right\}\omega=0,$$

with the conditional equation $\frac{d\mathrm{P}}{dx}+\frac{2i}{x}\mathrm{P}+\frac{h}{v}\mathrm{P}^2=-v$, instead of (3). This Riccatian can be reduced in a similar manner to that employed in $(\mathrm{A}_1,\ \mathrm{B}_1)$"].

8299. (By Asûtosh Mukhopâdhyây, B.A., F.R.A.S.)—Integrate the equation $\left(\frac{d\theta}{d\phi}\right)^2=\frac{\sin\theta}{\sin 2\phi}$.

Solution by Professor Sebastian Sircom, M.A.

Writing $\frac{1}{2}\pi-2\theta$ for θ, and $\frac{1}{2}\pi-\phi$ for ϕ, we have $\frac{2d\theta}{(\cos 2\theta)^{\frac{1}{2}}}\pm\frac{d\phi}{(\cos 2\phi)^{\frac{1}{2}}}=0.$

A particular integral of $\frac{d\psi}{(\cos 2\psi)^{\frac{1}{2}}} + \frac{d\phi}{(\cos 2\phi)^{\frac{1}{2}}} + \frac{d\mu}{(\cos 2\mu)^{\frac{1}{2}}} = 0$ is

$$\cos\psi = \cos\mu\cos\phi + \sin\mu\sin\phi\,(\cos 2\psi)^{\frac{1}{2}} \quad \ldots\ldots(1),$$

and, putting $\mu = \phi = \theta$, $\cos\psi = \cos^2\theta + \sin^2\theta\,(\cos 2\psi)^{\frac{1}{2}}$ $\ldots\ldots(2)$

is a particular integral of $\frac{d\psi}{(\cos 2\psi)^{\frac{1}{2}}} + \frac{2\,d\theta}{(\cos 2\theta)^{\frac{1}{2}}} = 0$, which can be easily verified.

Eliminating ψ between (2) and (1), which is the complete primitive of $\frac{d\phi}{(\cos 2\phi)^{\frac{1}{2}}} + \frac{d\psi}{(\cos 2\psi)^{\frac{1}{2}}} = 0$, and restoring the original values of θ and ϕ, we have the solution required. (1) and (2) give

$$\sqrt{2}\cos\psi = \cos\mu\,(1+\sin 2\phi)^{\frac{1}{2}} + \sin\mu\,(1-\sin 2\phi)^{\frac{1}{2}}\,(\cos 2\psi)^{\frac{1}{2}},$$

$$2\cos\psi = 1 + \sin\theta + (1-\sin\theta)\,(\cos 2\psi)^{\frac{1}{2}};$$

(2) agrees with the equation $z = \frac{2x(1-x^2)^{\frac{1}{2}}\,(1-k^2x^2)^{\frac{1}{2}}}{1-k^2x^4}$ in the Duplication Theory of Elliptic Functions.

7786. (By D. Biddle.)—Three trains in motion, A, B, C, occupy three parallel sets of rails, and their several lengths are 150, 130, and 100 yds. A and B are up-trains travelling at uniform speeds of 20 and 25 miles an hour respectively. B has overtaken A, and 35 seconds have passed since they began to overlap, when the head of C, a down train, passes the head of A. C is slackening speed, and comes to rest 55 seconds afterwards, and 350 yards from the point indicated. Find (1) the period of time during which all three trains are side by side, and give the positions of commencement and finish of such juxtaposition, supposing C to slacken speed at the same rate that a body projected vertically from the earth would if *in vacuo*, the distances traversed being, when measured from the point of rest, as the squares of the times occupied in traversing them; and (2) determine also the whole extent of permanent way covered at one time or other, during the transit, by the triple rectangular juxtaposed mass, as well as the maximum length of such mass, and the limits between which its variable centre oscillates.

Solution by the Proposer.

The speed of A $= 9\frac{7}{9}$ yards per second, and of B $= 12\frac{2}{9}$ yards per second. The gain of B on A $= 2\frac{4}{9}$ yards per second, making $85\frac{5}{9}$ yards in 35 seconds; therefore $150 - 85\frac{5}{9} = 64\frac{4}{9}$ yards = distance between B and C at given instant.

Let x = number of seconds before B and C meet, and the three trains come into juxtaposition; y = number of seconds before B and C are clear of each other; z = number of seconds before A and C are clear of each other. Then

$$12\tfrac{2}{9}x + 350\left(1 - \frac{(55-x)^2}{55^2}\right) = 64\tfrac{4}{9};$$

$$12\tfrac{2}{9}y + 350\left(1 - \frac{(55-y)^2}{55^2}\right) = 294\tfrac{4}{9}; \quad \text{and} \quad 9\tfrac{7}{9}z + 350\left(1 - \frac{(55-z)^2}{55^2}\right) = 250.$$

Whence $x = 2{\cdot}6147$, $y = 12{\cdot}5297$, $z = 11{\cdot}8279$;
therefore $z-x$ = duration of juxtaposition of all three trains = $11{\cdot}8279 - 2{\cdot}6147 = 9{\cdot}2132$ seconds. And $64\frac{4}{9} - 12\frac{2}{9}x$ = the distance of the commencement of such juxtaposition from the point where C meets A $= 32{\cdot}487$ yards. The distance of its finish from the same point $= 150 - 9\frac{7}{9}z = 34{\cdot}3495$ yards. Consequently the positions of commencement and finish are not 2 yards apart.

When C meets A, its speed is the greatest of the three; but, before it meets B, the last-mentioned has the advantage; for equality in speed between B and C is reached in 2·182539 seconds, whereas the two do not meet until 2·6147 seconds have elapsed, as we have seen. Consequently, the first movement of the variable centre will be upward with B, which will cover more ground above the point of commencing juxtaposition than C does below it, in a given time.

Two important epochs in the transit are those in which the head of C passes the tail of A, and the head of B passes the tail of C. Still measuring from the original instant when C meets A, let u = number of seconds before head of C reaches tail of A, and w = number of seconds before head of B reaches tail of C. Then

$$9\tfrac{7}{9}u + 350\left(1 - \frac{(55-u)^2}{55^2}\right) = 150, \text{ and } 12\tfrac{2}{9}w + 350\left(1 - \frac{(55-w)^2}{55^2}\right) = 164\tfrac{4}{9};$$

whence $u = 6{\cdot}9107$, and $w = 6{\cdot}8060$; consequently, the head of B reaches the tail of C before the head of C reaches the tail of A, and the maximum length of the triple mass is the entire length of C, viz., 100 yards. This is only maintained, however, for ·1047 seconds, when the head of C reaches the tail of A, and the mass instantly begins to shorten. $12\frac{2}{9}w - 64\frac{4}{9} = 18{\cdot}74$ = number of yards above *given point*, and $150 - 9\frac{7}{9}u = 82{\cdot}4287$ = number of yards below *given point*, to which the triple mass extends at one time or other during the transit, therefore $18{\cdot}74 + 82{\cdot}4287 = 101{\cdot}1687$ yards = the whole extent of permanent way involved in that respect.

As to the limits between which the centre of triple juxtaposition oscillates, we have already seen that, commencing 32·487 yards below the *given point*, it proceeds at first upwards: but, when the head of B reaches the tail of C, the centre of juxtaposition has only travelled 1·227 yards from its starting point, viz., to a point 31·26 yards below the *given point*. This is its limit in that direction. It at once turns, and without intermission descends the line, until the vanishing point of the triple juxtaposition is reached, 34·3495 yards below the *given point*. Consequently the centre varies its position during the transit over 3·0895 yards.

Geometrical Notes. *By* R. Tucker, M.A.

Through the mid-points P, Q, R of the median lines of the triangle ABC parallels are drawn to the sides of the triangle, viz., $\alpha_2 P\beta_1$, $\alpha_3 P\gamma_1$; $\beta_3 Q\gamma_2$, $\alpha_2 Q\beta_1$; $\alpha_3 R\gamma_1$, $\beta_3 R\gamma_2$. K_1, K_2, K_3 are the Symmedian-points respectively of the triangles $P\beta_1\gamma_1$, $Q\gamma_2\alpha_2$, $R\alpha_3\gamma_3$. Then the trilinear coordinates are, if $\kappa = a^2 + b^2 + c^2$,

$$K_1,\ a\Delta/\kappa,\ \tfrac{1}{4}a \sin C + b\Delta/\kappa,\ \tfrac{1}{4}a \sin B + c\Delta/\kappa,$$

with analogous values for K_2, K_3; it is easily seen that the coordinates of

the centroid of $K_1K_2K_3$ (K′) are $(3a^2+\kappa)\,\Delta/3a\kappa$, &c. Hence, since (*Quar. Jour. of Math.*, Vol. XX., No. 78, and *Reprint*, Vol. XLII., p. 25)

$$a\alpha\,(b^2-c^2)+\ldots+\ldots=0 \qquad \text{.....(i.)}$$

is the equation to the Symmedian-point axis, K′ is a point on this axis. The "α-coordinates" of the S.-point of $K_1K_2K_3=(a^2+\kappa)\,\Delta/2a\kappa$, therefore this point is also on (i.). If the median through A cuts K_2K_3 in g, and KM is perpendicular from K (the S.-point of ABC) on BC, then, since $K_3g = CM/4$, the S.-point of $K_1K_2K_3$ is on K_1g, and so on.

The projection of BK_1 on $BC = a\,(a^2+2c^2)/2K$, therefore (*Quar. Jour. of Math.*, Vol. XIX., No. 76, p. 345) the Brocard-centre of ABC is the orthocentre of $K_1K_2K_3$.

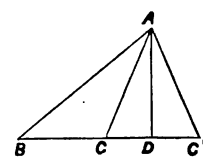
Again, it is not difficult to see that K is for $K_1K_2K_3$ a point exactly analogous to the point π of the triangle ABC (see *Quar. Jour.*, Vol. XX., No. 78, and *Educational Times Reprint*, *l.c.*).

The S.-point axis of $K_1K_2K_3$, and of the analogous triangles, is then coincident with the S.-point axis of the system considered in the articles just cited. This fact suggests numerous interesting results, which we leave to the reader.

3377. (By ARTEMAS MARTIN, M.A., F.R.A.S.)—The sum of two sides of a triangle is $2l$, and the perpendicular from the angle included by these sides upon the third side is p; find the average area of the triangle.

Solution by the Rev. T. C. SIMMONS, M.A.

Let ABC be a triangle, such that $AB+AC=2l$, and the altitude $AD=p$, which, if the triangle is to be possible, must be $<l$.



Then, considering AB to be the longer of the two sides, there will always be, corresponding to ABC, another triangle ABC′ satisfying the same conditions, and such that the sum of the areas of ABC, ABC′ is twice the area of ABD. We require, then, the average area of this latter triangle.

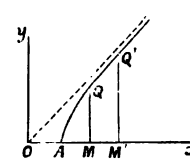
Put $AB=x$, AC or $AC'=2l-x$, then

$$\text{area of ABD} = \tfrac{1}{2}AD\,.\,BD = \tfrac{1}{2}p\,(x^2-p^2)^{\frac{1}{2}}.$$

Also, since l and p are respectively the maximum and minimum values of AC, the value of x ranges continuously between $2l-p$ and l. The required average area will consequently be represented by

$$\frac{p}{2\,(l-p)}\int_l^{2l-p}(x^2-p^2)^{\frac{1}{2}}\,dx,$$

an integral whose value in algebraical terms is cumbrous and uninviting; but which admits of the following simple geometrical interpretation. Draw the rectangular hyperbola whose semi-axis $OA=p$. Mark off the abscissa $OM=l$, also $MM'=MA$. Then, drawing the ordinates MQ, M′Q′, we at once see that the required average area is represented by $\frac{1}{2}OA/MM'$ times the area

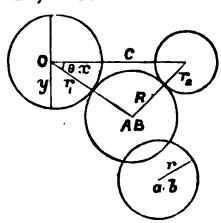


of MQQ'M'; or, which is the same thing, by the area of a triangle whose base is OA, and whose altitude is the mean of the ordinates situated between QM and Q'M'.

When $l = p$, or M and M' both move up to A, this triangle vanishes, and when $l = \infty$, its area becomes $\frac{1}{3}p \cdot \frac{1}{2}(l + 2l - p)$; results which *à priori* considerations, applied to the question, enable us easily to verify.

8371. (By Professor PURSER, M.A.)—A variable circle touches two fixed circles, the character of the contacts being given. Show that its chord of intersection with any third fixed circle envelopes a conic section.

Solution by ARTHUR HILL CURTIS, LL.D., D.Sc.



Let the equations of the two touched circles be

$$x^2 + y^2 = r_1^2, \quad (x-c)^2 + y^2 = r_2^2 \ldots\ldots (1, 2),$$

of the third fixed circle

$$(x-a)^2 + (y-b)^2 = r^2 \ldots\ldots\ldots\ldots (3),$$

and of the variable circle

$$(x-\mathrm{A})^2 + (y-\mathrm{B})^2 = \mathrm{R}^2 \ldots\ldots\ldots (4),$$

then if the centres of the circles (2) and (4) subtend the angle θ at the centre of the circle (1),

$$(\mathrm{R} + r_2)^2 = (\mathrm{R} + r_1)^2 + c^2 - 2c(\mathrm{R} + r_1)\cos\theta \ldots\ldots\ldots\ldots (5).$$

This is $2\mathrm{R} = \dfrac{2cr_1\cos\theta - m^2}{n - c\cos\theta}$, where $m^2 = r_1^2 - r_2^2 + c^2$, $n = (r_1 - r_2)$,

$$2\mathrm{A} = 2(\mathrm{R} + r_1)\cos\theta = \frac{(2nr_1 - m^2)\cos\theta}{n - c\cos\theta},$$

$$2\mathrm{B} = 2(\mathrm{R} + r_1)\sin\theta = \frac{(2nr_1 - m^2)\sin\theta}{n - c\cos\theta},$$

while $\mathrm{A}^2 + \mathrm{B}^2 - \mathrm{R}^2 = (\mathrm{R} + r_1)^2 - \mathrm{R}^2 = 2\mathrm{R}r_1 + r_1^2 = \dfrac{r_1(2cr_1\cos\theta - m^2)}{n - c\cos\theta} + r_1^2$.

We seek the envelope of $2\mathrm{A}x + 2\mathrm{B}y + \mathrm{R}^2 - \mathrm{A}^2 - \mathrm{B}^2 = 2ax + 2\,y + r^2 - a^2 - b^2$, or of

$$\frac{2nr_1 - m^2}{n - c\cos\theta}(x\cos\theta + y\sin\theta) - r_1\frac{(2cr_1\cos\theta - m^2)}{n - c\cos\theta} = 2ax + 2by + r^2 - r_1^2 - a^2 - b^2,$$

which is of the form $\mathrm{L}\cos\theta + \mathrm{M}\sin\theta = \mathrm{N}$, where L, M, N are of the first degree in x and y, and the envelope is $\mathrm{L}^2 + \mathrm{M}^2 = \mathrm{N}^2$, a conic.

8275, 8321, 8394. (By Prof. SYLVESTER, F.R.S., & T. MUIR, LL.D.)—Prove that

$$\begin{vmatrix} \lambda + aa + a'a' & ab + a'b' & ac + a'c', \ \ldots \\ \beta a + \beta' a' & \lambda + \beta b + \beta' b' & \beta c + \beta' c', \ \ldots \\ \gamma a + \gamma' a' & \gamma b + \gamma' b' & \lambda + \gamma c + \gamma' c', \ \ldots \end{vmatrix} = \lambda^{n-2}\{\lambda^2 + \lambda(\mathrm{A} + \mathrm{B}) + \mathrm{AB} - \mathrm{CD}\};$$

where n = order of determinant,

$A = (a, b, c, \ldots \Sigma a, \beta, \gamma, \ldots)$, $B = (a, \beta, \gamma, \ldots \Sigma a', b', c')$,

$C = (a, b, c, \ldots \Sigma a', \beta', \gamma', \ldots)$, $D = (a', b', c', \ldots \Sigma a, \beta, \gamma)$.

This result, when $A = s = B$ and C or D = 0, becomes $\lambda^{n-2}(\lambda + s)^2$.

Solution by Professor Sebastian Sircom, M.A.

$$\begin{vmatrix} aa+a'a', & ab+a'b', & ac+a'c', & \ldots \\ \beta a+\beta' a', & \beta b+\beta' b', & \beta c+\beta' c', & \ldots \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \end{vmatrix} = \begin{vmatrix} a, & a', & 0 \ldots \\ \beta, & \beta', & 0 \ldots \\ \gamma, & \gamma', & 0 \ldots \end{vmatrix} \begin{vmatrix} a, & a', & 0, & \ldots \\ b, & b', & 0, & \ldots \\ c, & c', & 0, & \ldots \end{vmatrix}$$

vanishes, and with it all its minors except those of the last two orders. Hence the given determinant must be of the form $\lambda^{n-2}(\lambda^2 + P\lambda + Q)$ and is the eliminant of

$$(aa+a'a')\,x + (ab+a'b')\,y + (ac+a'c')\,z + \ldots = -\lambda x, \text{ \&c.},$$

or of

$$a(ax+by+cz+\ldots) + a'(a'x+b'y+c'z+\ldots) = -\lambda x,$$

$$\beta(ax+by+cz+\ldots) + \beta'(a'x+b'y+c'z+\ldots) = -\lambda y, \text{ \&c.}$$

Multiplying by $a, b, c \ldots$ and adding, and by $a', b', c' \ldots$ and adding, we obtain

$$(aa+\beta b+\gamma c+\ldots+\lambda)\,X + (a'a+\beta' b+\gamma' c+\ldots)\,Y = 0,$$

and

$$(aa'+\beta b'+\gamma c'+\ldots)\,X + (a'a'+\beta' b'+\gamma' c'+\ldots+\lambda)\,Y = 0,$$

where X, Y are written for $(ax+by+cz+\ldots)$, $(a'x+b'y+c'z+\ldots)$, whence a factor of the eliminant of the system is $\lambda^2 + \lambda(A+B) + AB - CD$, where $A = (a, b, c \ldots \Sigma a, \beta, \gamma \ldots)$, $B = (a', b', c' \ldots \Sigma a', \beta', \gamma' \ldots)$,

$C = (a, b, c \ldots \Sigma a', \beta', \gamma' \ldots)$, $D = (a', b', c' \ldots \Sigma a, \beta, \gamma \ldots)$.

8432. (By Professor Crofton, F.R.S.)—Prove that

$$\Delta^n 0^{n+r} = n\,!\,.\,\Sigma(n+1)\,\Sigma(n+2)\ldots\Sigma(n+r),$$

the integration being with respect to n. Thus

$$\Delta^n 0^{n+1} = \frac{n(n+1)}{2\,!}\,n\,!, \quad \Delta^n 0^{n+2} = \frac{n(n+1)(n+2)(3n+1)}{4\,!}\,n\,!, \text{ \&c.}$$

Solution by H. Fortey, M.A.

Let $\phi(n, r) = \dfrac{\Delta^n 0^{n+r}}{n\,!}$, then we have

$\Delta\phi(n, r) = (n+1)\,\phi(n+1, r-1)$ (De Morgan's *Diff. Calc.*, p. 260),

therefore $\Delta\phi(n+1, r-1) = (n+2)\,\phi(n+2, r-2)$ &c., &c.;

$$\Delta\phi(n+r-2, 2) = (n+r-1)\,\phi(n+r-1, 1),$$

$$\Delta\phi(n+r-1, 1) = (n+r)\,\phi(n+r, 0) = n+r;$$

$\therefore\ \phi(n+r-1, 1) = \Sigma(n+r), \phi(n+r-2, 2) = \Sigma(n+r-1)\,\Sigma(n+r)$, &c. = &c.

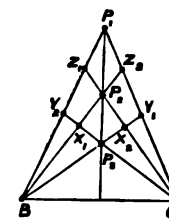
$$\phi(n, r) = \Sigma(n+1)\,\Sigma(n+2)\ldots\Sigma(n+r),$$

therefore $\Delta^n 0^{n+r} = n\,!\,\Sigma(n+1)\,\Sigma(n+2)\ldots\Sigma(n+r)$.

[This question will be found in BOOLE's *Finite Differences*, 2nd ed., p. 120, with a reference to HARGREAVE, *Quarterly Journal*, VIII., 26.]

8434. (By Professor NEUBERG.)—Sur une base donnée BC et d'un même côté, on construit six triangles semblables entre eux; démontrer que les sommets de ces triangles sont sur une même circonférence.

Solution by R. F. DAVIS, M.A.; D. BIDDLE; *and others.*

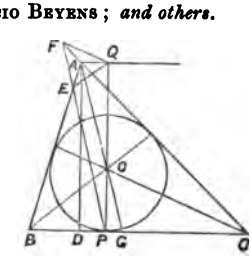


Let three isosceles triangles on the same side of a common base BC have their vertices at P_1, P_2, P_3, these points lying on the line bisecting the base at right angles, and let $P_1BC + P_2BC + P_3BC = \pi$.

Then, if X_1, X_2, Y_1, Y_2, Z_1, Z_2 denote the points of intersection of (P_2B, P_3C), (P_2C, P_3B), (P_3B, P_1C), (P_3C, P_1B), (P_1B, P_2C), (P_1C, P_2B), the triangles having BC for base, and these six points for apices, will be all similar to each other. These six points are symmetrically situated in pairs with respect to the line $P_1P_2P_3$; and a circle can be drawn having its centre on that straight line, which will pass through any two pairs of them, such as Y_1, Y_2, Z_1, Z_2; and since angle $Y_2X_1Z_2 = BX_1C = P_1CB = Y_2Y_1Z_2$, therefore X_1 lies on this circle; so similarly does X_2.

8399. (By the EDITOR.)—If from a point P in the base BC of a triangle ABC, PEF be drawn parallel to the bisector of the angle A, cutting AB, AC in E, F, and POQ be drawn through the in-centre O, cutting a parallel to BC through A in Q; prove that (1) AE=AF=AQ; and (2) if P be the point of contact of the in-circle with BC, and p the perpendicular from A on BC, then $PE \cdot PF = p^2$.

Solution by D. BIDDLE; Professor IGNACIO BEYENS; *and others.*



1. $\angle AFE = \frac{1}{2}A$, and $\angle AFE + AEF = A$; therefore $\angle AEF = AFE$, and $AE = AF$. Again, $BG : GP = AB : AE$, and $GP : OG = AQ : AO$; whence $AE = AB \cdot GP / BG$, and $AQ = AO \cdot GP / OG$. But $AB : BG = AO : OG$; therefore $AE = AQ = AF$.

2. In this case, $PQ = AD = p$. Moreover, $\angle QFA = \frac{1}{2}C$, $\angle AQE = \frac{1}{2}B$, and $\angle AFE = \frac{1}{2}A$; whence it is easy to see, $\angle EQP = EFQ$. Consequently, we have two triangles, PQE, PQF, having the angle at P in common, and $\angle PQE = PQF$. They are therefore similar. Hence $PE : PQ = PQ : PF$, or $PE \cdot PF = p^2$.

8436. (By Professor SIRCOM, M.A.)—A tricuspidal quartic passes through a fixed point, and has its cusps at fixed points; prove that the locus of its cuspidal centre is a conic touching the sides of the triangle of reference.

Solution by Prof. MATHEWS, M.A.; Prof. SARKAR, B.A.; *and others.*

Take the triangle formed by the cusps for the triangle of reference; the equation of the quartic will be

$$l^2y^2z^2 + m^2z^2x^2 + n^2x^2y^2 - 2mnx^2yz - 2nly^2zx - 2lmz^2xy = 0,$$

with a condition $A^2l^2 + B^2m^2 + C^2n^2 - 2BCmn - 2CAnl - 2ABlm = 0$.

The cuspidal tangents are

$$mz - ny = 0,\ nx - lz = 0,\ ly - mx = 0;$$

hence the cuspidal centre is $x/l = y/m = z/n$,
and its locus is the conic

$$A^2x^2 + B^2y^2 + C^2z^2 - 2BCyz - 2CAzx - 2ABxy = 0.$$

8035. (By Rev. T. C. SIMMONS, M.A.)—If three lengths be chosen at random, prove that they are just as likely as not to denote the sides of a possible triangle.

Solution by the PROPOSER.

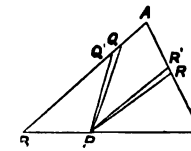
Of the three lengths, one (and each is equally likely) *must* be not less than either of the two others; denote it by l. Then all we know of the other lengths is that each lies between 0 and l. But if two lines are known to be of random lengths between 0 and l, it is an even chance that their sum is greater than l, and the required result follows.

[The PROPOSER remarks that "the question has been previously given (see Vol. IX., p. 45, and Vol. XXXIV., p. 22) by two mathematicians of high standing, who appear to be of opinion that the result only holds *when it is stated beforehand that the maximum limit for each length is the same.* The above mode of reasoning would, however, make it appear that the statement of this restriction is unnecessary. See also Vol. XLIV., p. 117."]

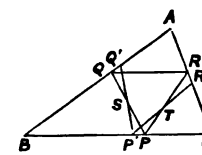
8400. (By Rev. J. J. MILNE, M.A.—Similar to the Editor's Quest. 8052.)—A point P is taken on the base of a triangle, and two lines including a given angle drawn through it so as to cut off from the triangle a quadrilateral of maximum area. Prove that (1), whatever be the position of P, the sides of the quadrilateral through P are equal to one another; and (2) in the position of P which gives the greatest maximum, those sides are equally inclined to the base of the triangle.

Solution by W. J. Barton, M.A.; Rev. H. London, M.A.; *and others.*

(1) Let PQAR be the maximum quadrilateral for the position P; let PR′, PQ′ be indefinitely near to PR, PQ, so that the angle Q′PR′ = the angle QPR; then the quadrilateral PQAR = the quadrilateral PQ′AR′; therefore the triangle PQQ′ = the triangle PRR′; and (since the angles QPQ′, RPR′ are equal) therefore PQ . PQ′ = PR . PR′. But, when Q′, R′ move up to, and coincide with Q, R, PQ = PQ′, PR = PR′; therefore $PQ^2 = PR^2$; therefore PQ = PR.



(2) Let PQAR be the greatest maximum quadrilateral, P′Q′AR′ another maximum quadrilateral, P′ being indefinitely near to P. Let PQ, P′Q′; PR, P′R′ intersect at S and T respectively; then, since angle SPT = angle SP′T, a circle can be described through P, P′, S, T. Now, the triangle BPQ = the triangle BP′Q′; therefore the triangle SQQ′ = the triangle SPP′; and therefore, as in (1), SQ = SP in the limit; therefore S becomes the middle point of PQ, and similarly T the middle point of PR; therefore ST is parallel to QR. Also, when P′ is indefinitely near to P, the circle STPP′ cuts BC in two consecutive points at P; *i.e.*, the circle about PST touches BC at P; therefore angle SPB = angle PTS; and angle TPC = angle TSP; and, since PS, PT are equal, being halves of PQ, PR, the angle PST = angle PTS; therefore angle QPB = angle RPC.



[This question was proposed in a wider form by the Editor 23 years ago as Quest. 1438, and two solutions of it (one geometrical, the other analytical), containing an elaborate discussion of its various cases, may be seen in our first volume. The theorems may be easily extended to the cases in which for a triangle there is substituted a polygonal or curvilinear area.]

8283. (By Capt. H. Brocard.)—La parabole tangente aux deux bissectrices d'un angle A du triangle et aux deux hauteurs correspondant aux côtés de cet angle a pour équation

$$(\beta \cos C + \gamma \cos B)^2 + 4a \cos A\,(a \cos A - \beta \cos B - \gamma \cos C) = 0.$$

Solution by Rev. T. Galliers, M.A.; Professor Sarkar, M.A.; *and others.*

The general equation of a conic is

$$u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0;$$

let $\quad vw - u'^2 = U, \quad wu - v'^2 = V, \quad uv - w'^2 = W,$

$$v'w' - uu' = U', \quad w'u' - vv' = V', \quad u'v' - ww' = W'.$$

The condition that the straight line (l, m, n) should touch the conic is shown in treatises on Trilinears to be

$$Ul^2 + Vm^2 + Wn^2 + 2U'mn + 2V'nl + 2W'lm = 0 \ldots\ldots\ldots\ldots (1),$$

and the condition that the conic should be a parabola (since in this case it touches the line at infinity) is

$$U \sin^2 A + V \sin^2 B + W \sin^2 C + 2U' \sin B \sin C + 2V' \sin C \sin A + 2W' \sin A \sin B = 0 \ldots\ldots (2.)$$

Expanding the equation given in the question, and comparing it with the general equation, we shall find that

$U = 0$, $V = 4 \cos^2 A (\cos^2 B - \cos^2 C)$, $W = 4 \cos^2 A (\cos^2 C - \cos^2 B)$,

$U' = 0$, $V' = 2 \cos C \cos A (\cos^2 C - \cos^2 B)$, $W' = 2 \cos A \cos B (\cos^2 B - \cos^2 C)$;

the equations to the bisectors of the angle A are $\beta \pm \gamma = 0$. Here $l = 0$, $m = 1$, $n = \pm 1$, and (1) reduces to $V + W \pm 2U' = 0$, an equation which holds since $V + W = 0$, and $U' = 0$. The equations of the perpendiculars from B and C on the opposite sides are respectively

$$-\cos A \,.\, \alpha + \cos C \,.\, \gamma = 0, \quad \cos A \,.\, \alpha + (-\cos B)\, \beta = 0.$$

Hence the condition (2) becomes for these straight lines respectively

$$U \cos^2 A + W \cos^2 C - 2V' \cos A \cos C = 0,$$

$$U \cos^2 A + V \cos^2 B - 2W' \cos A \cos B = 0,$$

and, putting for U, V, W, V', W' their values given above, we see that these equations are satisfied.

Lastly, since $U = 0$, $U' = 0$, $V + W = 0$, the condition (2) becomes, when we omit the common factor $4 \cos A (\cos^2 B - \cos^2 C)$,

$$\cos A (\sin^2 B - \sin^2 C) - \cos C \sin C \sin A + \cos B \sin B \sin A = 0.$$

Now, $\sin^2 B - \sin^2 C = \sin (B + C) \sin (B - C) = \sin A \sin (B - C)$,

and $\sin B \cos B - \sin C \cos C = \frac{1}{2} (\sin 2B - \sin 2C)$

$$= \sin (B - C) \cos (B + C) = -\sin (B - C) \cos A.$$

Hence the above equation becomes the identity

$$\cos A \sin A \sin (B - C) - \cos A \sin A \sin (B - C) = 0.$$

Hence the condition (1) is satisfied, and thus the given equation is that of a parabola satisfying the specified conditions.

7831. (By Asútosh Mukhopádhyáy, B.A.)—Show that

$$u \equiv \sin \theta + \tfrac{1}{3} \sin 3\theta + \tfrac{1}{5} \sin 9\theta + \ldots \text{ ad inf.} = \tfrac{1}{4} \log \frac{1 + \tan \frac{1}{2}\theta}{1 - \tan \frac{1}{2}\theta}.$$

Solution by B. Hanumanta Rau, M.A.; Belle Easton; *and others.*

$$\frac{du}{d\theta} = \cos \theta + \cos 3\theta + \cos 9\theta + \ldots$$

$$= \tfrac{1}{4} \operatorname{cosec} 2\theta \{ \sin 3\theta + \sin \theta + (\sin 7\theta - \sin 3\theta) + (\sin 11\theta - \sin 7\theta) + \ldots \}$$

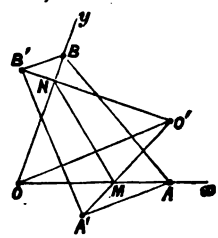
$$= \tfrac{1}{4} \sin \theta \operatorname{cosec} 2\theta = \tfrac{1}{8} \sec^2 \tfrac{1}{2}\theta;$$

therefore $u = \int \frac{d\theta}{4 \cos \theta} = \frac{1}{4} \int \frac{2 \cdot d \tan \frac{1}{2}\theta}{1 - \tan^2 \frac{1}{2}\theta} = \frac{1}{4} \log \frac{1 + \tan \frac{1}{2}\theta}{1 - \tan \frac{1}{2}\theta}$.

No constant is required, as u and θ vanish together.

8366. (By W. S. M'Cay, M.A.)—If two triangles have the same centre of gravity and have a centre of perspective at infinity, prove that their axis of perspective passes through their common centre of gravity.

Solution by R. F. Davis, M.A.; Emily Perrin; *and others.*



Let OAB, O'A'B' be the two triangles having OO', AA', BB' parallel to each other; and a common centre of gravity, so that OO' = AA' + BB'. Let also OA, O'A' intersect in M, and OB, O'B' in N; then MN is the axis of perspective. Take OA, OB as axes; OA = a, OB = b, OM = α, ON = β. Then, since

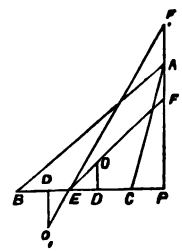
$$AA' / OO' + BB' / OO' = 1,$$

therefore $AM / OM + BN / ON = 1$,

or $a/\alpha + b/\beta = 3$. Wherefore MN, whose equation is $x/\alpha + y/\beta = 1$, passes through the centre of gravity whose coordinates are $3x = a$ and $3y = b$.

8266. (By Captain MacMahon, R.A. Generalised in Quest. 8400.)—Prove that the straight lines joining the mid-point of the side BC of the triangle ABC with the in-centre and the ex-centre to BC, meet the perpendicular from A on BC, each in a point distant from A a length equal to the radius of the corresponding circle.

Solution by W. J. Greenstreet, B.A.; R. Knowles, B.A.; *and others.*



Let O, O_1 be the centres of in- and ex-circles; let E be the mid-point of BC; and let OE, O_1E meet the perpendicular AP in F, F_1 respectively, then drawing OD, O_1D_1 perpendicular to BC, and putting r for the in-radius, we have

$$AF = AP - PF = \frac{2\Delta}{a} - \frac{OD \cdot PE}{ED}$$

$$= \frac{2\Delta}{a} - \frac{r(EC + CP)}{BD - BE} = \frac{2\Delta}{a} - \frac{r(\frac{1}{2}a - b\cos C)}{(s-b) - \frac{1}{2}a}$$

$$= \frac{2\Delta}{a} - r\left(\frac{1}{2}a - \frac{a^2+b^2-c^2}{2ab}.b\right) \times \frac{2}{c-b}$$

$$= \frac{2\Delta}{a} - r\left(\frac{c^2-b^2}{2a}\right)\left(\frac{2}{c-b}\right) = \frac{2rs}{a} - \frac{r(c+b)}{a} = \frac{r}{a}[2s-(b+c)] = r;$$

$$AF_1 = PF_1 - PA = \frac{r_1 PE}{ED_1} - \frac{2\Delta}{a} = r_1 \frac{c^2-b^2}{2a} \cdot \frac{2}{c-b} - \frac{2\Delta}{a},$$

because $ED_1 = \frac{1}{2}a - BD_1 = \frac{1}{2}a - (s-c) = \frac{1}{2}(c-b)$

$$= r_1\left(\frac{c+b}{a} - \frac{2(s-a)}{a}\right) = r_1.$$

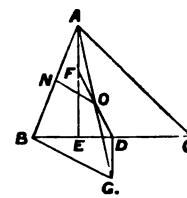
[If O be the in-centre, and D the mid-point of the side BC, we have

AF : AO = DG : GO, and GO = GB;

hence DG : GB = AF : AO,

and the triangles AON, BGD are similar, thus we have DG : GB = ON : AO,

therefore AF = ON. A similar construction and proof would hold for the ex-centre.]



8048. (By Professor Cochez.)—Trouver la somme

$$\frac{1}{\cos x \cos y} + \frac{1}{\cos y \cos z} + \ldots \frac{1}{\cos t \cos u} + \frac{1}{\cos u \cos w},$$

$x, y, z \ldots t, u, w$ étant en progression arithmétique, ρ étant la raison.

Solution by W. J. Curran Sharp, M.A.

Since $\sin(y-x) = \sin y \cos x - \cos y \sin x$,

therefore $\sin(y-x) \sec x \sec y = \tan y - \tan x$;

similarly $\sin(z-y) \sec y \sec z = \tan z - \tan y$, &c., &c.,

$\sin(w-u) \sec u \sec w = \tan w - \tan u$ and $y-x = z-y = \&c. = \rho$;

$\therefore$ by addition $\sin\rho \,.\, S = \tan w - \tan x$ or $S = (\tan w - \tan x) / \sin\rho$.

8404. (By Maurice D'Ocagne.)—La courbe décrite par le centre de gravité des points de rencontre d'une conique fixe et d'un cercle mobile, de rayon nul, est transformée linéaire de la courbe décrite par le centre de ce cercle.

Solution by R. F. Davis, M.A.; Rev. T. Galliers, M.A.; *and others.*

Let $b^2x^2 + a^2y^2 - a^2b^2 = 0$ and $(x-\alpha)^2 + (y-\beta)^2 = 0$ be the equations of the conic and circle. Eliminating y, we get for the abscissæ of the points of intersection

$(a^2-b^2)^2 x^4 - 4a^2\alpha\,(a^2-b^2)\,x^3 + \ldots = 0$; hence $x_1 + x_2 + x_3 + x_4 = 4a^2\alpha/(a^2-b^2)$.

Thus $\bar{x} = a^2\alpha / (a^2-b^2)$; similarly $\bar{y} = b^2\beta / (b^2-a^2)$.

8418. (By W. J. C. Sharp, M.A.)—If the sides of a spherical triangle ABC be bisected in the points D, E, F, show that

$$\cos \mathrm{DE} \sec \tfrac{1}{2}c = \cos \mathrm{EF} \sec \tfrac{1}{2}a = \cos \mathrm{FD} \sec \tfrac{1}{2}b.$$

Solution by R. F. Davis, M.A.; Prof. Ignacio Beyens, M.A.; *and others.*

Since $\cos c = \cos a \cos b + \sin a \sin b \cos C$,

$\therefore\ 1 + \cos a + \cos b + \cos c = (1 + \cos a)(1 + \cos b) + \sin b \sin c \cos C$

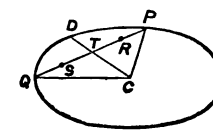
$= 4 \cos \frac{1}{2}a \cos \frac{1}{2}b \left\{\cos \frac{1}{2}a \cos \frac{1}{2}b + \sin \frac{1}{2}a \sin \frac{1}{2}b \cos C\right\} = 4 \cos \frac{1}{2}a \cos \frac{1}{2}b \cos DE,$

and $\cos DE \sec \frac{1}{2}c$ = a symmetrical quantity. [Hence if one of the arcs of great circles which join the mid-points of the sides of a spherical triangle be a quadrant, then the other two are also quadrants.]

8317. (By Professor Sáradáranjan Ráy, M.A.)—The triangle formed by a chord of a given ellipse and the radii vectores drawn from the centre to its extremities is given in area; show that the locus of a point that divides the chord in a fixed ratio is a concentric and similar ellipse.

Solution by Rev. H. London, M.A.; Rev. T. Galliers, M.A.; *and others.*

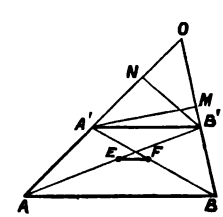
Let PQ be the chord, and R the point of section. Make QS = PR, then S is a point on the locus. Draw CD a diameter conjugate to PQ. Then, by orthogonal projection, the proposition follows at once. For the area of triangle CRT, and the ratio PT : TR, are both constant.



8362. (By Maurice d'Ocagne.)—Si sur chacune des diagonales d'un trapèze prise pour diamètre on décrit un cercle, l'axe radical de ces cercles est perpendiculaire aux bases du trapèze et passe par le point de rencontre des côtés obliques.

Solution by R. F. Davis, M.A., *and* Prof. Ignacio Beyens.

Let ABB′A′ be the trapezium having AB, A′B′ for parallel sides (bases); and let E, F be the mid-points of AB′, A′B. Then EF is parallel to and equidistant from AB, A′B′; and the radical axis of the circles upon AB′, A′B as diameters is perpendicular to these parallels. Produce AA′, BB′ to meet in O, and draw A′M, B′N perpendiculars on OB, OA. Then ON . OA′ = OM.OB′, or ON . OA = OM.OB, and O is therefore a point on the radical axis.



[The third diagonal of the quadrilateral meets the two parallel sides at infinity, hence the circle drawn on it as diameter becomes a straight line coaxal with the circles on the remaining diagonals as diameters.]

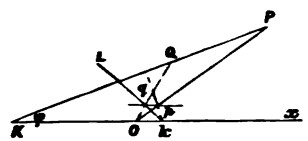
8306. (By Professor SYLVESTER, F.R.S.)—If the product of $\rho(\cos\lambda + i\sin\lambda) - r(\cos\alpha + i\sin\alpha)$ by $\frac{1}{\rho}(\cos\lambda + i\sin\lambda) - r(\cos\alpha + i\sin\alpha)$, is $P(\cos\omega + i\sin\omega)$, find the value of ω by a geometrical construction.

Solution by EMILY PERRIN; A. GORDON; *and others.*

Let the angle $POx = \alpha$, $OP = r$, $Op = 1/r$, $\angle QOx = \lambda$, $OQ = \rho$, $Oq = 1/\rho$; then

$\rho(\cos\lambda + i\sin\lambda) - r(\cos\alpha + i\sin\alpha) = PQ$,

$\frac{1}{\rho}(\cos\lambda + i\sin\lambda) - \frac{1}{r}(\cos\alpha + i\sin\alpha) = pq$.



If, now, PQ cut the initial line Ox in K, and qp cut Ox in k, PQ . pq turns pq through an angle = PKx, and if kL be drawn making the angle qkL = angle PKx, angle Lkx = ω. Now, angle Lkx = sum of angles QOx, Oqk, Lkq, = sum of angles α, POx, OPQ, because the triangles Oqp, OPQ are similar, but the angles POx, OPQ together = λ, therefore angle ω = angle Lkx = $\alpha + \lambda$.

[Assuming DE MOIVRE'S theorem

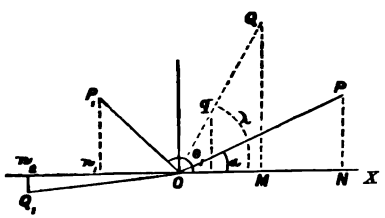
$A(\cos\alpha + i\sin\alpha)B(\cos\beta + i\sin\beta)$

$= AB[\cos(\alpha+\beta) + i\sin(\alpha+\beta)]$

(or that the product of two quaternions of this form = a third quaternion), let O be the origin of coordinates; take

XOP = α, XOQ = λ, OP = r,

OQ = ρ, $Oq = \rho^{-1}$;



draw the ordinates as in the figure to P, Q, q; take

$On_1 = OM - ON = \rho\cos\lambda - r\cos\alpha$, $On_2 = \rho^{-1}\cos\lambda - r\cos\alpha = Om - ON$,

$n_1P = MQ - NP = \rho\sin\lambda - r\sin\alpha$, $n_2Q_1 = \rho^{-1}\sin\lambda - r\sin\alpha = mq - NP$;

and let $P_1Ox = \theta_1$, $Q_1Ox = \theta_2$; then the angle required is $\theta_1 + \theta_2 = \omega$, where, in the formula for A, B, α, β, we have OP_1, OQ_1; hence

$$\tan\omega = \tan(\theta_1 + \theta_2) = \frac{\rho\sin 2\lambda - r(1+\rho^2)\sin(\alpha+\lambda) + \rho r^2\sin 2\alpha}{\rho\cos 2\lambda - r(1+\rho^2)\cos(\alpha+\lambda) + \rho r^2\cos 2\alpha}.$$

7829. (By J. BRILL, B.A.)—If ABC be a plane triangle, and O a point within it; prove that

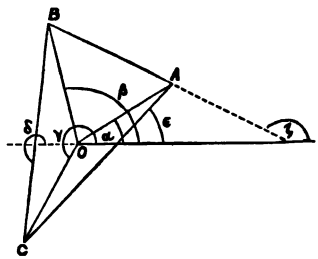
(1) $OB \cdot OC \cdot BC\cos(OBA - OCA) + OC \cdot OA \cdot CA\cos(OCB - OAB) + OA \cdot OB \cdot AB\cos(OAC - OBC) = BC \cdot CA \cdot AB$.

(2) $OA^2 \cdot BC \cdot \cos(OAB - OAC) + OB^2 \cdot CA\cos(OBC - OBA) + OC^2 \cdot AB\cos(OCA - OCB) = BC \cdot CA \cdot AB$.

Solution by the PROPOSER.

(1) Let

$(OA) = a(\cos\alpha + \iota\sin\alpha)$,
$(OB) = b(\cos\beta + \iota\sin\beta)$,
$(OC) = c(\cos\gamma + \iota\sin\gamma)$,
$(BC) = d(\cos\delta + \iota\sin\delta)$,
$(CA) = e(\cos\epsilon + \iota\sin\epsilon)$,
and $(AB) = f(\cos\zeta + \iota\sin\zeta)$.



Since $(BC) = (OC) - (OB)$,
$(CA) = (OA) - (OC)$,
and $(AB) = (OB) - (OA)$,

we have, by means of a well-known algebraical theorem,

$$(OB).(OC).(BC) + (OC).(OA).(CA) + (OA).(OB).(AB) + (BC).(CA).(AB) = 0.$$

Hence, we obtain

$$bcd\cos(\beta+\gamma+\delta) + cae\cos(\gamma+\alpha+\epsilon) + abf\cos(\alpha+\beta+\zeta) + def\cos(\delta+\epsilon+\zeta) = 0,$$

and $$bcd\sin(\beta+\gamma+\delta) + cae\sin(\gamma+\alpha+\epsilon) + abf\sin(\alpha+\beta+\zeta) + def\sin(\delta+\epsilon+\zeta) = 0.$$

From these two equations we easily deduce

$$bcd\cos(\beta+\gamma-\epsilon-\zeta) + cae\cos(\gamma+\alpha-\zeta-\delta) + abf\cos(\alpha+\beta-\delta-\epsilon) + def = 0.$$

Now, $\beta-\zeta = -OBA$, and $\gamma-\epsilon = \pi + OCA$, therefore

$$\cos(\beta+\gamma-\epsilon-\zeta) = \cos(\pi - OBA + OCA) = -\cos(OBA - OCA);$$

and in a similar manner we can prove that

$$\cos(\gamma+\alpha-\zeta-\delta) = -\cos(OCB - OAB),$$

and $$\cos(\alpha+\beta-\delta-\epsilon) = -\cos(OAC - OBC).$$

Thus we finally obtain

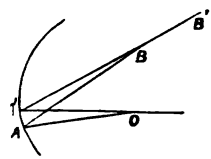
$$OB.OC.BC\cos(OBA-OCA) + OC.OA.CA\cos(OCB-OAB) + OA.OB.AB\cos(OAC-OBC) = BC.CA.AB.$$

(2) In a similar manner we can deduce the second result from the equation $(OA)^2.(BC) + (OB)^2.(CA) + (OC)^2.(AB) + (BC).(CA).(AB) = 0$; or it may be deduced from the first result by inverting the figure with respect to the point O.

8281. (By Professor BYOMAKESA CHAKRAVARTI, M.A.)—A traveller in a railway carriage moving on a curve in a flat country is seated with his face towards the engine, and looks out on the inside of the curve. Show that the country beyond a certain point disappears, and that the country nearer than this point comes into view; and that this point moves with a velocity $\frac{d\rho}{dt}\sin\alpha + v\cos\alpha$, where ρ is the radius of curvature of the curve, v the velocity of the train, and α the angle which the plane from

his eye to the vertical edge, nearest the engine, of the window which bounds his view makes with the direction of the train's motion.

Solution by Rev. J. J. Milne, M.A.; J. O'Regan; *and others.*



Let AO, A'O be two consecutive normals. AB, A'B' bounds the view in the two cases so that all *behind* AB can be seen. If these intersect in B, it is evident that B is the point required, for the country ABA' disappears, and BB' comes into view.

Let $AOA' = d\phi$, $ABA' = d\chi$. Then, since the angle $BA'O = BAO$, a circle will go round BA'AO. Therefore $d\phi = d\chi$. Let $AB = u$.

Then $ud\chi$ = perpendicular from A' on $AB = AA' \sin a = \rho\, d\phi \sin a$;

Therefore $u = \rho \sin a$, therefore $A'B' = u + du = (\rho + d\rho) \sin a$;

but $BB' = A'B' + AA' \cos a - AB$, therefore $BB' = d\rho \sin a + ds \cos a$,

therefore, dividing by dt, the velocity of $B = \dfrac{d\rho}{dt} \sin a + v \cos a$.

8290. (By J. Brill, M.A.)—Prove that in any triangle (n positive integer)

$$c^n = a^n \cos nB + na^{n-1} b \cos[(n-1)B - A] + \frac{n(n-1)}{2!} a^{n-2} b^2 \cos[(n-2)B - 2A] + \ldots + b^n \cos nA \ldots\ldots (1);$$

$$\frac{c^n}{b^n} \cos nA = 1 + n \cdot \frac{a}{c} \cos B + \frac{n(n+1)}{2!} \cdot \frac{a^2}{c^2} \cos 2B + \ldots \text{where } a < c \ldots (2).$$

Solution by Rev. J. J. Milne, M.A.; Rev. T. Galliers, M.A.; *and others.*

1. Let $C = a^n \cos nB + na^{n-1} b \cos[(n-1)B - A] + \ldots\ldots,$

$S = a^n \sin nB + na^{n-1} b \sin[(n-1)B - A] + \ldots\ldots,$

therefore $C + Si = a^n e^{nBi} + na^{n-1} b \cdot e^{[(n-1)B - A]i} + \ldots\ldots$

$= (ae^{Bi} + be^{-Ai})^n = \{a \cos B + b \cos A + i(a \sin B - b \sin A)\}^n = c^n;$

therefore $C = c^n, \quad S = 0.$

2. Let $C = 1 + n \dfrac{a}{c} \cos B + \ldots\ldots, \quad S = n \dfrac{a}{c} \sin B + \ldots\ldots;$

therefore $C + Si = 1 + n \dfrac{a}{c} \cdot e^{Bi} + \dfrac{n(n+1)}{2!} \dfrac{a^2}{c^2} e^{2Bi} + \ldots\ldots$

$= \left(1 - \dfrac{a}{c} e^{Bi}\right)^{-n} = c^n (c - a \cos B - i \cdot a \sin B)^{-n}$

$= c^n (b \cos A - i\, b \sin A)^{-n} = \dfrac{c^n}{b^n} (\cos nA + i \sin nA);$

therefore $C = \dfrac{c^n}{b^n} \cos nA; \quad S = \dfrac{c^n}{b^n} \sin nA.$

8291. (By W. J. McClelland, B.A.)—If a triangle ABC be inscribed in a circle of a co-axal system, the three sides each touching circles of the system at points X, Y, Z; prove that the lines AX, BY, CZ are concurrent.

Solution by the Proposer.

Let P be a limiting point, then

$$\frac{PB}{PC} = \frac{BX}{CX}, \quad \frac{PC}{PA} = \frac{CY}{AY}, \quad \frac{PA}{PB} = \frac{AZ}{BZ};$$

hence $$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = 1;$$ therefore, &c.

8352. (By Professor Sáradáranjan Ráy, M.A.)—One angle A of a triangle is given in magnitude and position; if the base a also be given, show that the bisectors of the exterior base angles touch their envelopes at distances $2a \cos B \cos \frac{1}{2}C \operatorname{cosec} A$ and $2a \cos C \cos \frac{1}{2}B \operatorname{cosec} A$ from the respective base angles.

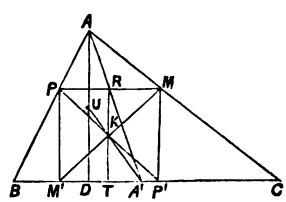
Solution by R. F. Davis, M.A.; Charlotte A. Scott, B.Sc.; *and others.*

Let B'C' and B'P be consecutive positions of the side BC and the external bisector BP of B. Then in the triangle BPB' the angle $BPB' = \frac{1}{2}\delta B = -\frac{1}{2}\delta C$ (since B + C is constant), and $BB' = \delta c$. Hence $BP : BB' :: \sin \frac{1}{2}(\pi - B) : \frac{1}{2}\delta C$, or $BP = 2 \cdot \frac{\delta c}{\delta C} \cos \frac{1}{2}B$. But, if $k \sin A = a$ we have $k \sin C = c$, and $\delta c = \delta C \cdot k \cos C$; hence $BP = 2k \cos \frac{1}{2}B \cos C = 2a \operatorname{cosec} A \cos \frac{1}{2}B \cos C$.

8388. (By R. F. Davis, M.A.)—If ABC be a triangle, and AD, BE, CF perpendiculars upon the opposite sides, the lines joining the middle points of (AD, BC), (BE, CA), (CF, AB), each pass through the symmedian point.

Solution by Profs. Ignacio Beyens *and* Byomakesa Chakravarti.

Soit K le centre des symedianes; nous savons (voir Casey's *Sequel to Euclid*, p. 163) que les trois droites antiparallèles MM', NN', PP', par rapport aux trois côtés BA, BC, CA du triangle qui passent par K, sont egales et sont divisées en deux parties égales par le même point, et MPM'P', NN'MM' sont des rectangles dont le centre est K.



Cela posé: la médiane AA' passera par

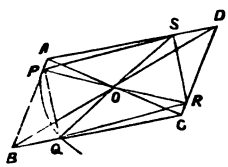
le milieu de MP, la droite KR = KT sera parallèle à AD et la droite A'U, que dans le triangle AA'D joigne le vertex A' avec le milieu U du côté opposé AD, coupera à RT en K qui est le centre des symedianes. De la même manière on démontrera pour les côtés AB, AC et les hauteurs qui le correspondent.

[Captain BROCARD remarks that "cette question a été traitée plusieurs fois: M. E. LEMOINE, dans sa note *Sur les propriétés du centre des Médianes antiparallèles* (v. Association française pour l'avancement des Sciences; Congrès de Lille, séance du 22 Août 1874), l'a enoncée. Dans son "*Etude sur le Cercle de Brocard*," (v. Journal de Math. Elém., 1883, § 28), M. Auguste Morel en donne une démonstration. Enfin elle se trouve mentionné dans ma note sur "*les droites et points inverses* (v. Journal de Math. Elém. 1885.]

8316. (By Professor CLARKE, F.R.S.)—Inscribe the maximum rectangle in a given parallelogram.

Solution by Rev. H. LONDON, M.A.; R. KNOWLES, B.A.; *and others.*

Let O be the centre of the parallelogram. Draw OP perpendicular to AB. With centre O describe a circle PQ touching AB at P. Draw POR, QOS. Then PQRS is the maximum rectangle. For it is a rectangle, all the angles being those of a semicircle. And it is a maximum, since, if any point P' be taken in AB close to P, the triangle QP'S = triangle QPS.

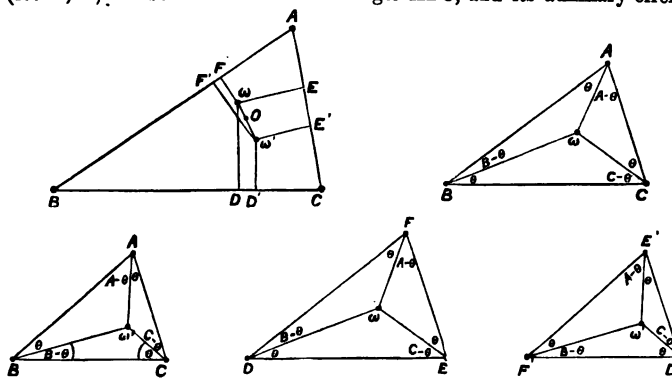


8375. (By Rev. T. C. SIMMONS, M.A.)—If from ω, ω', the respectively *positive* and *negative* Brocard-points of a triangle ABC (whose Brocard-angle is θ), perpendiculars be drawn on the sides, thereby forming two pedal triangles DEF, D'E'F', then (1) these triangles are similar to ABC and equal to one another; prove also (2) that they have a common circumcircle, the triangular coordinates of whose centre, lying midway between ω and ω', are proportional to $a^2(b^2+c^2)$, $b^2(c^2+a^2)$, $c^2(a^2+b^2)$; (3) that ω is the *positive* Brocard-point of DEF, and the centre of similitude of ABC, DEF; likewise ω' the *negative* Brocard-point of D'E'F'; (4) that the other Brocard-points of DEF, D'E'F' lie on the circle described on $\omega\omega'$ as diameter; (5) that the angles which the sides of DEF, D'E'F' make with corresponding sides of ABC are all equal to $\frac{1}{2}\pi - \theta$; (6) that the ratio of similitude of the triangles DEF, ABC, or of D'E'F', ABC is equal to $\sin\theta$; (7) that the triangles AFE', BDF', CED' are all similar to ABC, and the lines DE', EF', FD' respectively parallel to AB, BC, CA; (8) that the intercepts DD', EE', FF' on the sides are respectively proportional to $\sin C \sin(C-A) + \sin B \sin(B-A)$, and two similar expressions; (9) that if R be the circumradius, and ρ the "T. R." radius of ABC, and if R' be the circumradius of DEF or D'E'F', then $R'^2/R^2 + R^2/4\rho^2 = 1$.

[In the absence of a better means of distinguishing between the Brocard-points ω, ω' of a triangle ABC, Mr. SIMMONS suggests the following:—Joining ω to B, suppose the Brocard-angle given by ωBC. Then, if in moving from ω to B and therefrom to C we revolve round the angle ωBC in what is known as the positive direction, call ω the *positive* Brocard-point; if in a negative direction, call ω the *negative* Brocard-point. The above circle, if thought worthy of a name, may be called the *sine-circle* of the triangle, since its radius = R sin θ.]

Solution by R. F. DAVIS, M.A.

Since ω, ω' are the Brocard-points of the triangle ABC, the angles ωBC, ωCA, ωAB, ω'CB, ω'AC, ω'BA are each equal to θ. Hence an ellipse (foci ω, ω') can be inscribed in the triangle ABC, and its auxiliary circle



(whose centre is O, the middle point of $\omega\omega'$) will pass through the six points D, E, F, D′, E′, F′. This ellipse is the "Brocard-ellipse" of the triangle. From the different tetrads of concyclic points, we have the angles

$$\left.\begin{matrix}\omega EF = \omega FD = \omega DE \\ \omega'F'E' = \omega'D'F' = \omega'E'D'\end{matrix}\right\} = \theta,$$

$$E\omega F = E'\omega'F' = \pi - A,\quad F\omega D = F'\omega'D' = \pi - B,\quad D\omega E = D'\omega'E' = \pi - C.$$

(1, 2) The triangles FDE, E′F′D′ are each similar to the triangle ABC, and therefore to each other. Since they are inscribed in the same circle they are moreover equal to one another.

The trilinear coordinates of O are proportional to $\frac{1}{2}(b/c + c/b)$ or to $a(b^2 + c^2)$, &c.

(3) Also ω is the *positive* Brocard-point of the triangle FDE. Since the triangles ABC, FDE are similar and have a common positive Brocard-point ω, this latter point is the centre of similitude.

Similarly ω' is the *negative* Brocard-point of the triangle E′F′D′.

(4) Since the triangle E′F′D′ can be turned about O so as to come into the position of the triangle FDE; it is obvious that, if ρ, ρ' be the other Brocard-points, then $O\rho = O\rho' = O\omega = O\omega'$, and ρ, ρ' lie on the circle described on $\omega\omega'$ as diameter.

(5) DE, BC are corresponding sides of the similar triangles FDE, ABC and the angle between them $= \frac{1}{2}\pi - \angle \omega DE = \frac{1}{2}\pi - \theta$.
Similarly for triangles E'F'D', ABC.

(6) Ratio of similitude of triangles FDE, ABC is $\omega D : \omega B$ or $\sin\theta$. Similarly, &c.

(7) Since $\angle AFE' = \pi - \angle F'FE' = F'D'E' = C$; hence FE' is an anti-parallel to BC and triangles AFE', ACB are similar.
Since $\angle E'DD' = \angle E'F'D' = B$, DE' is parallel to AB.

(8) DD' : EE' : FF' are as the sines of the angles subtended at the circumference. Now DD' subtends an angle

$$CD'E' - CDE' = \tfrac{1}{2}\pi - (C - \theta) - B = \theta + A - \tfrac{1}{2}\pi,$$

and is proportional to $\cos(\theta + A)$; $\cos(\theta + A)$ is again proportional to

$$\cos A \cot\theta - \sin A = \cos A (\cot A + \cot B + \cot C) - \sin A,$$

or to $$\sin 2A \sin A + 2 \cos 2A \sin B \sin C$$

$$= \sin B \sin(2A + C) + \sin C \sin(2A + B) = \sin B \sin(B - A) + \sin C \sin(C - A).$$

(9) From (6) we have $R' = R \sin\theta$; but $R = 2\rho \cos\theta$ (Casey), whence, &c.

8363. (By H. G. Dawson, B.A.)—Solve the equations

$$x^2 - yz - zw - wy = a^2, \quad y^2 - zx - xw - wz = b^2 \ldots\ldots\ldots\ldots (1, 2),$$
$$z^2 - xy - yw - wx = c^2, \quad w^2 - xy - yz - zx = d^2 \ldots\ldots\ldots\ldots (3, 4).$$

Solution by F. R. J. Hervey, M.A.; Professor Mahendra Nath Ray, M.A., LL.B.; *and others.*

Let $P \equiv x + y + z + w$; then $(x - y) P = a^2 - b^2$, $(x - z) P = a^2 - c^2$, &c...(5), whence $4x = P + (3a^2 - b^2 - c^2 - d^2) / P$; with similar expressions for $4y$, &c. It remains to express P. Adding the given equations, as they stand, and after squaring, we have

$$\Sigma a^2 = \Sigma x^2 - 2\Sigma xy = 2\Sigma x^2 - P^2 ; \quad (\Sigma x^2)^2 = \Sigma a^4 ;$$

whence $$P^2 = -\Sigma a^2 \pm 2\sqrt{\Sigma a^4}, \text{ and } x = \frac{2a^2 - \Sigma a^2 \pm \sqrt{\Sigma a^4}}{2\left\{-\Sigma a^2 \pm 2\sqrt{\Sigma a^4}\right\}^{\frac{1}{2}}}.$$

The upper sign gives real values, since $4\Sigma a^4 - (\Sigma a^2)^2 = \Sigma (a^2 \sim b^2)^2$.

[From (5), $$\frac{x-y}{a^2-b^2} = \frac{x-z}{a^2-c^2} = \frac{x-w}{a^2-d^2} = \frac{1}{\rho} \text{ (say)},$$

$\therefore \quad \rho x - a^2 = \rho y - b^2 = \rho z - c^2 = \rho w - d^2 = \sigma$ say, $\quad \therefore \quad x = \dfrac{\sigma + a^2}{\rho}$, $y =$ &c.;

therefore, from (5), $$\rho^2 = a^2 + b^2 + c^2 + d^2 + 4\sigma ;$$

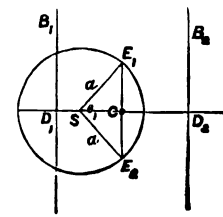
hence, from (1), $$2\sigma^2 + 2\sigma L + M = 0,$$

where $L \equiv a^2 + b^2 + c^2 + d^2$ and $M = a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2$.
There will be but one value of σ if $a^4 + b^4 + c^4 + d^4 = 0$.]

8397. (By Professor HUDSON, M.A.)—If the orbits of two comets (supposed parabolic) intersect the orbit of the earth (supposed circular) in the same two given points, prove that, if t_1, t_2 be the times in which the comets respectively move from one of these points to the other, and a year be the unit of time, $(t_1+t_2)^{\frac{2}{3}}+(t_1-t_2)^{\frac{2}{3}}=(4/3\pi)^{\frac{2}{3}}$.

Solution by Dr. CURTIS, LL.D., D.Sc.; Rev. T. GALLIERS, M.A.; *and others.*

Let the circle in the figure, of radius a, represent the Earth's orbit; E_1, E_2 the two given points, subtending at S, the centre of the circle, the angle $2\theta_1$; SC the bisector of this angle; D_1 and D_2 the points on SC, such that $CD_1=CD_2=a$; then D_1B_1, D_2B_2, drawn at right angles to SC, will be the directrices of the two parabolic orbits. Let the parameters of these be $4m_1$, $4m_2$; then, if μ denote the mass of the sun,



$$t_1(2\mu m_1)^{\frac{1}{2}}=2\int_0^{\theta_1}r^2\,d\theta=2\int_0^{\theta_1}\frac{m_1^2\,d\theta}{\cos^4\frac{1}{2}\theta}=4m_1^2\int_0^{\theta_1}(1+\tan^2\tfrac{1}{2}\theta)\,d\tan\tfrac{1}{2}\theta$$
$$=4m_1^2(\tan\tfrac{1}{2}\theta_1+\tfrac{1}{3}\tan^3\tfrac{1}{2}\theta_1).$$

Hence we have $t_1=\dfrac{4m_1^{\frac{2}{3}}}{(2\mu)^{\frac{1}{2}}}(\tan\frac{1}{2}\theta_1+\frac{1}{3}\tan^3\frac{1}{2}\theta_1)$, but $a=\dfrac{m}{\cos^2\frac{1}{2}\theta_1}$,

therefore $m_1^{\frac{2}{3}}=a^{\frac{2}{3}}\cos^3\frac{1}{2}\theta_1$, while τ, the periodic time of the earth, $=\dfrac{2\pi a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}$;

therefore $$t_1=\frac{2^{\frac{1}{2}}\tau}{3\pi}(3\cos^2\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_1+\sin^3\tfrac{1}{2}\theta_1);$$

similarly, changing θ_1 into $\pi-\theta_1$, $t_2=\dfrac{2^{\frac{1}{2}}\tau}{3\pi}(3\sin^2\frac{1}{2}\theta_1\cos\frac{1}{2}\theta_1+\cos^3\frac{1}{2}\theta_1)$,

therefore $$(t_1+t_2)=\frac{2^{\frac{1}{2}}\tau}{3\pi}(\sin\tfrac{1}{2}\theta_1+\cos\tfrac{1}{2}\theta_1)^3,$$

and $$(t_1-t_2)=\frac{2^{\frac{1}{2}}\tau}{3\pi}(\sin\tfrac{1}{2}\theta_1-\cos\tfrac{1}{2}\theta_1)^3;$$

therefore $$(t_1+t_2)^{\frac{2}{3}}+(t_1-t_2)^{\frac{2}{3}}=\left(\frac{4}{3\pi}\right)^{\frac{2}{3}}$$

which, taking τ as the unit of time, reduces to the formula proposed.

8241. (By D. EDWARDES.)—Show that

$$\left(\frac{1}{1+n^2}+\frac{1}{3^2+n^2}+\frac{1}{5^2+n^2}+\&c.\right)\left(\frac{1}{n^2}+\frac{2}{2^2+n^2}+\frac{2}{4^2+n^2}+\&c.\right)=\frac{\pi^2}{8n^2}.$$

Solution by A. GORDON; J. BRILL, M.A.; *and others.*

If we develope $e^{a(\pi-x)}+e^{-a(\pi-x)}$ in a series of cosines, we find (TODHUNTER, *Integral Calculus*)

$$\frac{\pi}{2a}\,\frac{e^{a(\pi-x)}+e^{-a(\pi-x)}}{e^{a\pi}-e^{-a\pi}} = \frac{1}{2a^2}+\frac{\cos x}{1^2+a^2}+\frac{\cos 2x}{2^2+a^2}+\dots$$

(true between $x = 0$, $x = \pi$ inclusive).

Substituting for x, $x = 0$ and $x = \pi$, we have two series, A and B ; if we multiply (A + B) by (A − B) and reduce, we obtain the required result.

8213. (By R. Knowles, B.A.)—If tangents TP, TQ from T (x, y) touch the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ in P, Q, show that

$$\text{TP}.\text{TQ} = \frac{(a^2y^2+b^2x^2-a^2b^2)\,[(x^2+y^2)^2+(a^2-b^2)(a^2-b^2-2x^2+2y^2)]^{\frac{1}{2}}}{a^2y^2+b^2x^2}.$$

Solution by W. J. Greenstreet, B.A. ; E. Rutter ; *and others.*

$$\Delta\text{TPQ} = \tfrac{1}{2}\text{TP}.\text{TQ}\sin\phi = \text{quad. TPCQ} - \Delta\text{PCQ}$$

$$= \frac{(b^2h^2+a^2k^2-a^2b^2)^{\frac{3}{2}}}{(b^2h^2+a^2k^2)\,(b^2h^2+a^2k^2)},$$

therefore

$$\text{TP}.\text{TQ} = \frac{2\,(1+\tan^2\phi)^{\frac{1}{2}}}{\tan\phi}\cdot\frac{(b^2h^2+a^2k^2-a^2b^2)^{\frac{3}{2}}}{k^2h^2+a^2k^2}$$

$$= \frac{[(h^2+k^2-a^2-b^2)^2+4\,(b^2h^2+a^2k^2-a^2b^2)]^{\frac{1}{2}}}{(b^2h^2+a^2k^2-a^2b^2)^{\frac{1}{2}}}\cdot\frac{(b^2h^2+a^2k^2-a^2b^2)^{\frac{3}{2}}}{b^2h^2+a^2k^2}$$

= stated result.

8478. (By Professor Steggall, M.A.)—If l, m, n ; l', m', n' ; $\cos\alpha$, $\cos\beta$, $\cos\gamma$, be the direction cosines of the incident ray, the refracted ray, and the normal to the refracting surface at the point of incidence, prove that

$$\frac{l-\mu l'}{\cos\alpha} = \frac{m-\mu m'}{\cos\beta} = \frac{n-\mu n'}{\cos}.$$

Solution by W. W. Taylor, M.A. ; Rev. T. R. Terry, M.A. ; *and others.*

Let us take point of incidence as origin, and let $Ax + By + Cz = 0$....(1) be the plane which contains the two rays and the normal ; then

$$Al + Bm + Cn = 0,\quad Al' + Bm' + Cn' = 0,\quad A\cos\alpha + B\cos\beta + C\cos\gamma = 0 \qquad \text{.....(2, 3, 4).}$$

Let

$$A'x + B'y + C'z = 0 \qquad \text{.....(5)}$$

be the plane at right angles to (1) passing through the normal, then

$$AA' + BB' + CC' = 0,\quad A'\cos\alpha + B'\cos\beta + C'\cos\gamma = 0 \qquad \text{.....(6, 7),}$$

and the condition of refraction is that the perpendicular from the point l, m, n on (2) $= \mu$ perpendicular from the point l', m', n' on (2) ; or $A'l + B'm + C'n = \mu\,(A'l' + B'm' + C'n')$, which may be written thus,

$$A'(l-l'\mu) + B'(m-m'\mu) + C'(n-n'\mu) = 0 \qquad \text{.....(8).}$$

Eliminating A′, B′, C′ from (6), (7), (8) we get (9), and multiplying in (9) first column by A, second by B, and third by C, and adding for a new third column, then using relations (2), (3), (4), we get (10):

$$\begin{vmatrix} A, & B, & C \\ \cos\alpha, & \cos\beta, & \cos\gamma \\ l-l'\mu, & m-m'\mu, & n-n'\mu \end{vmatrix} = 0, \qquad \begin{vmatrix} A & B & A^2+B^2+C^2 \\ \cos\alpha & \cos\beta & 0 \\ l-l'\mu & m-m'\mu & 0 \end{vmatrix} = 0$$

.....(9, 10),

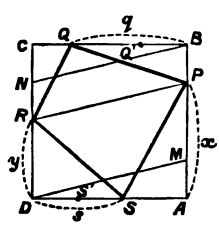
therefore $\frac{l-l'\mu}{\cos\alpha} = \frac{m-m'\mu}{\cos\beta}$ = by symmetry $\frac{n-n'\mu}{\cos\gamma}$, [or by (7) and (8)].

[Let O be the origin of coordinates, and let OI, OR be unit lengths measured along the incident ray and the refracted ray; and let ON, the normal cut IR in N. Then the coordinates of I, R are l, m, n; l', m', n' respectively; and NI : NR :: μ : 1 by elementary trigonometry. Hence the coordinates of N are $l-\mu l'$, $m-\mu m'$, $n-\nu n'$; and these are proportional to $\cos\alpha$, $\cos\beta$, $\cos\gamma$, whence the result.]

8318 & 8444. (By the Editor.)—If four points P, Q, R, S be taken at random, one in each side of a quadrilateral ABCD, find (1) the probability that the area of PQRS will be less than n (area of ABCD), where n is a proper fraction; (2) apply the result to the case of a rectangle; and therefrom (3) show that the probability that PQRS will be less than three-fourths (or, what is the same thing, greater than one-fourth) of ABCD is

$$\tfrac{1}{8} + \tfrac{5}{4}\log_e 2 = \cdot 991434.$$

Solution by Rev. T. C. Simmons, M.A.



First, let ABCD be a square, each of whose sides = 1; and let the distances of the random points from the angles be given respectively by

$$AP = x,\ BQ = q,\ DR = y,\ DS = s.$$

We may, without loss of generality, consider only those cases in which $x > y$. Draw BN, DM parallel to PR, then

$$\begin{aligned} \text{area of PQRS} &= \Delta PRQ + \Delta PRS \\ &= (\Delta BPN + q \text{ times } \Delta BNC) \\ &\qquad + (MRP + s \text{ times } ADM) \\ &= \tfrac{1}{2}BM \cdot AD + \tfrac{1}{2}(q+s)\,AM \cdot AD \\ &= \tfrac{1}{2}(1-x+y) + \tfrac{1}{2}(q+s)(x-y). \end{aligned}$$

Hence $$\text{twice PQRS} = 1 + (q+s-1)(x-y).$$

When this = 2, we must have $x = 1$, $y = 0$, $q = 1$, $s = 1$; when it = 0, we must have $x = 1$, $y = 0$, $q = 0$, $s = 0$, results which are also manifest from inspection. When it $= 2n$ (n being a fraction between 0 and $\frac{1}{4}$), then $x - y = \frac{1-2n}{1-q-s}$, showing that the least value of $x-y$ is $1-2n$, and that, for a quadrilateral less than n times ABCD, the limits of x are 1 and $1-2n$, and those of y are $x-1+2n$ and 0.

Moreover, in the same case, $q+s=1-\frac{1-2n}{x-y}$, showing that, for a quadrilateral less than n times ABCD, q must lie between $1-\frac{1-2n}{x-y}$ and 0, and s between $1-q-\frac{1-2n}{x-y}$ and 0.

Hence the required probability for this case is (putting $1-2n=\mu$)

$$\int_\mu^1\int_0^{x-\mu}\int_0^{1-\mu+(x-y)}\left(1-q-\frac{\mu}{x-y}\right)dx\,dy\,dq+\int_0^1\int_0^x\int_0^1 dx\,dy\,dq$$

$$=\int_\mu^1\int_0^{x-\mu}\left(1-\frac{\mu}{x-y}\right)^2dx\,dy=\int_\mu^1\left(x+2\mu\log\frac{\mu}{x}-\frac{\mu^2}{x}\right)dx$$

$$=\tfrac{1}{2}-\tfrac{1}{2}\mu^2+4n\mu+(3\mu^2+4\mu n)\log\mu \quad\ldots\ldots(1).$$

The above formula, investigated for $n<\frac{1}{3}$, can be utilised for $n>\frac{1}{3}$ as follows. On BC, AD, take BQ′ = CQ, and DS′ = AS. Then, if H, K be the mid-points of BC, AD, we evidently have

$$\triangle PQR+PQ'R+PSR+PS'R=2PHR+2PKR=\text{area of ABCD}.$$

So that, for every two points Q, S, giving with P, R a quadrilateral exceeding the half of ABCD by any amount, we have two corresponding points Q′, S′, giving with P, R a quadrilateral falling short of the half of ABCD by the same amount. Thus the chance that PQRS is less than the n^{th} of ABCD is the same as the chance that PQRS is greater than $(1-n)^{\text{th}}$ of ABCD. If, therefore, n be $>\frac{1}{3}$, all we have to do is to substitute $1-n$ for n in (1), and subtract the result from unity.

It has been hitherto assumed that ABCD is a square. But, since a square can be orthogonally projected into a parallelogram of any given species, and that in this case areas are projected in the same ratio, and also subdivisions of parallel lines in some other same ratio, it is easily seen that the above result holds for any kind of *parallelogram*.

In the case stated in the question, $n=\frac{1}{4}$, $\mu=\frac{1}{2}$, and (1) gives for the chance that PQRS is less than one-fourth of ABCD

$$p_1=\tfrac{1}{8}-\tfrac{5}{4}\log 2.$$

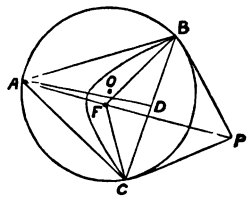
Hence the chance that PQRS is *greater* than one-fourth of ABCD is

$$p_2\equiv 1-p_1=\tfrac{1}{8}+\tfrac{5}{4}\log 2=\cdot 991434 \text{ nearly}.$$

[The Editor's solution of this problem was given in the year 1863, under Question 1282 (see Vol. I., 2nd ed., pp. 93—95), but, the correctness of the result having been called in question, Mr. Simmons has, at the Editor's request, devised an independent method for the sake of verification, though he states he "prefers the solution in Vol. I."]

8409. (By Émile Vigarié.)—On mène deux tangentes AB, AC à une parabole de foyer F et l'on circonscrit un cercle de centre O au triangle ABC. Démontrer que, si P est le pôle de BC par rapport à ce cercle: (1) la droite AP est médiane antiparallèle (symédiane) du triangle ABC; (2) la droite AP passe par le foyer F de la parabole; (3) l'angle AFO est droit; (4) les points O, P, B, C sont sur un cercle de diamètre OP qui passe par le point F.

Solution by R. F. Davis, M.A.; W. J. Barton, M.A.; *and others.*



Let BP, CP be tangents to the circle at B, C; and let D be the middle point of BC so that AD is parallel to the axis.

(1) Since BP = CP, sin BAP : sin CAP = sin ABP : sin ACP = sin C : sin B; therefore AP is the symmedian line through A.

(2) Since the triangles CFA, AFB are similar, the angle CAF = ABF = BAD, and AF coincides with AP.

(3) Since the angles OBP, OCP are right angles, and angle BOC = 2A = BFC; therefore B, F, C lie on the circle upon OP as diameter.

(4) And angle AFO = OFP = a right angle.

Note on Question 7831. *By* Professor Catalan.

La solution de cette *question* est *inadmissible*, parce que les séries considérées sont indéterminées. Mais ce n'est pas tout: la formule énoncée me paraît *fausse*. Voici pourquoi.

Dans mon *Traité élémentaire des séries* (Paris, 1860), je trouve (p. 105) ces deux formules, dues (peut-être) à Lobatto:

$$x\sin\phi-\tfrac{1}{3}x^3\sin 3\phi+\tfrac{1}{5}x^5\sin 5\phi-\ldots=\frac{1}{4}\,\mathrm{L}.\,\frac{1+2x\sin\phi+x^2}{1-2x\sin\phi+x^2}\ldots\ldots\ (1),$$

$$x\sin\phi+\tfrac{1}{3}x^3\sin 3\phi+\tfrac{1}{5}x^5\sin^5\phi+\ldots=\tfrac{1}{2}\arctan\frac{2x\sin\phi}{1-x^2}\ldots\ldots\ldots\ldots\ (2),$$

lesquelles supposent $x\lessgtr 1$, $\phi>0$, $\phi<\frac{1}{2}\pi$. Il en résulte, pour $x=1$:

$$\sin\phi+\tfrac{1}{3}\sin 5\phi+\tfrac{1}{5}\sin 9\phi+\ldots=\tfrac{1}{8}\mathrm{L}.\,\frac{1+\sin\phi}{1-\sin\phi}+\tfrac{1}{8}\pi\ldots\ldots\ldots\ldots\ (3).$$

Suivant M. Asûtosh Mukhopâdhyây ..., le second membre devrait être $\frac{1}{h}\mathrm{L}.\,\frac{1+\mathrm{tg}\,\frac{1}{2}\phi}{1-\mathrm{tg}\,\frac{1}{2}\phi}$. Je n'ai pas le temps de vérifier les formules (1), (2); mais j'ai lieu de croire qu'elles sont exactes.

Remarque.—Si, dans la formule (3), on supposait $\phi=0$, on trouverait ce résultat *absurde*, $0=\frac{1}{8}\pi$. C'est ce qui arrive, presque toujours, pour les séries trigonométriques.

8396. (By F. Purser, M.A.)—A variable conic circumscribes a given triangle, the normals at the vertices meeting in a point P. Show that the locus of P is a cubic passing through the vertices of the triangle, the circumcentre, orthocentre, incentre, and three excentres. The circumcentre is also the centre of the cubic, and the lines joining it to the middle points of the sides of the triangle are the three asymptotes.

Solution by Professor SIRCOM, M.A.; R. F. DAVIS, M.A.; *and others.*

Let $L\beta\gamma + M\gamma\alpha + N\alpha\beta = 0$ be the trilinear equation of any conic circumscribing the given triangle, and α', β', γ' be the coordinates of P. The equation to the tangent at A is $\beta/M + \gamma/N = 0$, and if this be perpendicular to $\beta/\beta' - \gamma/\gamma' = 0$, we have $N : M = 1 + \gamma' \cos A/\beta' : \cos A + \gamma'/\beta'$. Similarly, L : N and M : L. Eliminating L, M, N, and suppressing dashes, we get the locus

$$(\beta + \gamma \cos A)(\gamma + \alpha \cos B)(\alpha + \beta \cos C) = (\beta \cos A + \gamma)(\gamma \cos B + \alpha)(\alpha \cos C + \beta).$$

This cubic is a remarkable one. From its form, it passes through the vertices of the given triangle. By trial, it is found, the circumcentre (cos A, cos B, cos C), orthocentre (sec A, sec B, sec C), incentre (1, 1, 1), and excentres (−1, 1, 1, &c.) all lie on the curve.

If we seek the points of intersection of the straight line joining A with the circumcentre ($\beta \cos C - \gamma \cos B = 0$), we find at these points $\beta = 0$, $\alpha \cos B - \beta \cos A = 0$, and $\alpha + \beta \cos C = 0$. The last equation shows that the cubic passes through the extremity A′ of the diameter of the circumcircle through A. Similarly, it passes through B′, C′, and the circumcentre is its centre.

If we seek the points of intersection of the perpendicular through A on BC ($\beta \cos B - \gamma \cos C = 0$), we find at these points $\beta = 0$, $\alpha \cos A - \beta \cos B = 0$, and $\alpha \cos C + \beta = 0$. The last equation represents a line through C perpendicular to BC; and shows that one of the directions at infinity is perpendicular to BC.

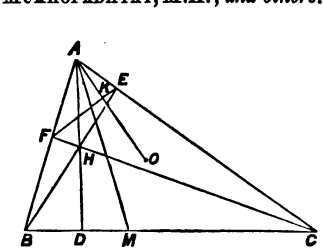
[From the form of the cubic it is obvious that it is its own inverse with respect to the triangle. It may, in fact, be also defined as the locus of a point (α, β, γ), such that the line joining it to its inverse $(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$ passes through the fixed point (cos A − cos B cos C, cos B − cos C cos A, cos C − cos A cos B).]

8446. (By Captain H. BROCARD.)—On construit (α) la parabole tangente à deux côtés du triangle et aux deux hauteurs correspondantes, (β) la parabole tangente aux pieds des bissectrices de l'angle compris entre ces deux côtés; démontrer (1) que ces deux coniques admettent pour foyer le pied de la hauteur partant du sommet de l'angle considéré; (2) determiner leurs divers éléments et évaluer leurs paramètres; et (3) fixer la situation de leurs points d'intersection.

Solution by R. F. DAVIS, M.A.; ASÛTOSH MUKHOPÂDHYÂY, M.A.; *and others.*

Let ABC be the triangle; AD, BE, CF its perpendiculars cointersecting in the orthocentre H. Then (1) the parabola touching AB, AC, BE, CF must have its focus at D, as the circumcircles of the triangles formed by any three of these four tangents all pass through D. Its directrix (locus of intersection of perpendicular tangents) will be EF, and its latus rectum

= 4 perpendicular from D on EF
= 4R sin 2B sin 2C.



(2) If AM, AN (meeting BC in M, N) be the internal and external bisectors of the angle A, then the parabola touching these perpendicular straight lines at M, N must have its focus at the foot of the perpendicular from A on MN (or BC), that is, the point D, while A is a point on its directrix. If O be the circumcentre, then, since AM bisects the angle between AD and AO, AO is the directrix and the latus rectum

$$= 4AD \sin (B-C) = 8R \sin B \sin C \sin (B-C).$$

(3) Let AO and EF, which are obviously perpendicular, intersect in K. Then the points of intersection of the two parabolas are the centres of the two circles which can be described so as to touch AO, EF and pass through D. They are, therefore, the intersection of the bisector of the right angle OKF with a circle having for extremities of a diameter the points dividing KD internally and externally in the ratio $\sqrt{2} : 1$.

8267. (By Asûtosh Mukhopâdhyây, M.A., F.R.A.S.)—Evaluate in a finite form, $\int_D^{-na} \frac{r\,dr}{\left[2a(n-1)+\{r^2-4a^2(n-1)\}^{\frac{1}{2}}\right]^3}$.

Solution by W. J. Curran Sharp, M.A.

If $\frac{r^2}{4a^2(n-1)^2} - \frac{1}{n-1}$ be put equal to $(u-1)^2$,

and therefore $\frac{r\,dr}{4a^2(n-1)} = (u-1)\,du$,

in the indefinite integral $\int \frac{r\,dr}{\left[2a(n-1)+\{r^2-4a^2(n-1)\}^{\frac{1}{2}}\right]^3}$,

this becomes $2a(n-1)\int \frac{(u-1)}{u^3}\,du = 2a(n-1)\left(-\frac{1}{u}+\frac{1}{2u^2}\right)$

$$= a(n-1)\frac{1-2u}{u^2} = 4a^2(n-1)^2 \frac{-a(n-1)-\left[r^2-4a^2(n-1)\right]^{\frac{1}{2}}}{\left[2a(n-1)+\{r^2-4a^2(n-1)\}^{\frac{1}{2}}\right]^3},$$

and the value of the definite integral is immediately obtained by substitution.

8309. (By Professor Neuberg.)—Soient $2a'$, $2b'$ deux diamètres d'une ellipse, perpendiculaires à deux diamètres conjugués. Démontrer que le rapport $\frac{a^2b^2-a'^2b'^2}{a^2+b^2-(a'^2+b'^2)}$ est constant.

Solutions by (1) *the* Proposer; (2) R. Knowles, B.A.; Rev. T. Galliers, M.A.; *and others.*

1. On a $a''^2+b''^2 = a^2+b^2$, $\frac{1}{a'^2}+\frac{1}{a''^2} = \frac{1}{a^2}+\frac{1}{b^2}$, $\frac{1}{b'^2}+\frac{1}{b''^2} = \frac{1}{a^2}+\frac{1}{b^2}$.

Entre ces égalités, il faut éliminer a'', b''. On a, d'abord,

$$1\Big/\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{a'^2}\right)+1\Big/\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{b'^2}\right)=a^2+b^2,$$

ou $$2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-\left(\frac{1}{a'^2}+\frac{1}{b'^2}\right)$$
$$=(a^2+b^2)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{a'^2}-\frac{1}{b'^2}\right)+\frac{a^2+b^2}{a'^2b'^2},$$

ou $$\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{a'^2}-\frac{1}{b'^2}\right)\left\{1-(a^2+b^2)\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right\}=(a^2+b^2)\left(\frac{1}{a'^2b'^2}-\frac{1}{a^2b^2}\right);$$

donc

$$\left(\frac{1}{a'^2b'^2}-\frac{1}{a^2b^2}\right)\Big/\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{a'^2}-\frac{1}{b'^2}\right)=\frac{a^2b^2-a'^2b'^2}{a'^2b'^2(a^2+b^2)-a^2b^2(a'^2+b'^2)}=\text{const.}$$

Prenons l'inverse et ajoutons a^2+b^2; on aura l'expression donnée ci-dessus.

2. *Otherwise*:—Let $a^2y^2+b^2x^2=a^2b^2$ be the equation to the ellipse, centre C, and $x'y'$ the coordinates of any point P; the equation to CP is $y=\frac{y'}{x'}x$, and that of a diameter perpendicular to it $y=-\frac{x'}{y'}x$; and, substituting in the equation to the ellipse, we get

$$x^2=\frac{a^2b^2y'^2}{a^2x'^2+b^2y'^2},\quad y^2=\frac{a^2b^2x'^2}{a^2x'^2+b^2y'^2},$$

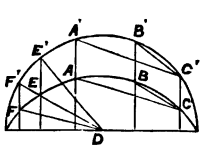
therefore $a'^2=\frac{a^2b^2(x'^2+y'^2)}{a^2x'^2+b^2y'^2}$, similarly $b'^2=\frac{a^2b^2(b^4x'^2+a^4y'^2)}{b^6x'^2+a^6y'^2}$,

and the ratio $\frac{a^2b^2-a'^2b'^2}{a^2+b^2-(a'^2+b'^2)}$ becomes $\frac{a^2b^2(a^8-a^6b^2-a^2b^6+b^8)x'^2y'^2}{(a^{10}-a^8b^2-a^2b^8+b^{10})x'^2y'^2}$, and is therefore constant.

8430. (By R. Curtis, M.A.)—A and B are two fixed points on an ellipse, C a variable point, D the centre, DE, DF parallels to BC, BA; prove that the area of the sector DEF is constant.

Solution by J. O'Byrne Croke, M.A.; W. J. Barton, M.A.; *and others.*

The ellipse may be considered the orthogonal projection of the auxiliary circle; hence C′, B′, A′, E′, F′ project into C, B, A, E, F, B′B, C′C, &c. being parallel to the minor axis. Thus the sector DE′F′ projects into the sector DEF; also the angle B′C′A′ is constant, and hence E′DF′ is constant; therefore the area of the sector DE′F′ is constant, as is also the area of the sector DEF.



8381 & 8448. (By JOHN GRIFFITHS, M.A.)—(8381.) If n be a number > 1 and s a number from 1 to $n-1$, show that

$$1 + 2\Sigma\,(-)^s \cos^2 s\pi / 2n = 0.$$

(8448.)—If n be an odd prime number and s a number from 1 to $\frac{1}{2}(n-1)$, show that $(-)^{\frac{1}{2}(n-1)}\, n \sec n\theta = \sec\theta \left\{1 + 2\Sigma \dfrac{(-)^s \cos(s\pi/n)}{1-\sin^2(s\pi/n)\sec^2\theta}\right\}$.

Solutions by D. EDWARDES; Rev. T. R. TERRY, M.A.; *and others.*

8381. Suppose n even. Then we have

$$(-)^{\frac{1}{2}n} \cos n\theta = 1 - \frac{n^2}{2!}\cos^2\theta + \frac{n^2(n^2-2^2)}{4!}\cos^4\theta - \&c.$$

Put $\cos n\theta = 0$, then θ may have any of the values $\dfrac{\pi}{2n}, \dfrac{3\pi}{2n}, \ldots \dfrac{(2n-1)\pi}{2n}$.

Also $$\cos\frac{(2n-1)\pi}{2n} = -\cos\frac{\pi}{2n},$$

$$\cos\frac{(2n-3)\pi}{2n} = -\cos\frac{3\pi}{2n}, \ldots\ldots \cos\frac{(n+1)\pi}{2n} = -\cos\frac{(n-1)\pi}{2n}.$$

Hence, if $x = \cos^2\theta$, the roots of

$$1 - \frac{n^2}{2!}x + \frac{n^2(n^2-2^2)}{4!}x^2 - \&c. = 0,$$

an equation of the $\frac{1}{2}n^{\text{th}}$ degree, are $\cos^2\dfrac{\pi}{2n}, \cos^2\dfrac{3\pi}{2n}, \ldots\ldots \cos^2\dfrac{(n-1)\pi}{2n}$.

And these roots are all different. Hence

$$\cos^2\frac{\pi}{2n} + \cos^2\frac{3\pi}{2n} + \ldots\ldots + \cos^2\frac{(n-1)\pi}{2n} = \tfrac{1}{4}n \quad \ldots\ldots\ldots\ldots (1).$$

Again, when n is even, we have

$$(-)^{\frac{1}{2}n+1} \sin n\theta = n \sin\theta \cos\theta \left\{1 - \frac{n^2-2^2}{3!}\cos^2\theta + \frac{(n^2-2^2)(n^2-4^2)}{5!}\cos^4\theta - \ldots\right\}.$$

If θ have any one of the values

$$\frac{\pi}{n}, \frac{2\pi}{n}, \ldots\ldots \frac{\frac{1}{2}n-1}{n}\pi, \frac{\frac{1}{2}n+1}{n}\pi, \ldots\ldots \frac{n-1}{n}\pi, \sin n\theta = 0.$$

And $$\cos\frac{n-1}{n}\pi = -\cos\frac{\pi}{n},$$

$$\cos\frac{n-2}{n}\pi = -\cos\frac{2\pi}{n}, \ldots\ldots \cos\frac{\frac{1}{2}n+1}{n}\pi = -\cos\frac{\frac{1}{2}n-1}{n}\pi.$$

Hence the equation $$-1 - \frac{n^2-2^2}{3!}x + \frac{(n^2-2^2)(n^2-4^2)}{5!}x^2 - \&c. = 0,$$

which is of the $(\frac{1}{2}n-1)^{\text{th}}$ degree, has the $\frac{1}{2}n-1$ different roots

$$\cos^2\frac{\pi}{n}, \cos^2\frac{2\pi}{n}, \ldots\ldots \cos^2\frac{\frac{1}{2}n-1}{n}\pi.$$

Hence $$\cos^2\frac{\pi}{n} + \cos^2\frac{2\pi}{n} + \cos^2\frac{3\pi}{n} + \ldots\ldots + \cos^2\frac{(\frac{1}{2}n-1)\pi}{n} = \tfrac{1}{4}(n-2) \ldots (2).$$

In the same way (1) and (2) may be shown to hold if n be odd, and Question 8381 follows by subtracting (1) from (2).

[For a first transformation, the formula $y = \sin(L + A + B + \ldots)$ gives

$$M\frac{\lambda}{k^2}\operatorname{sn}(Mu, \lambda) = \frac{\operatorname{sn} u \operatorname{cn} u}{\operatorname{dn} u}\left\{1 + 2\Sigma\frac{(-)^s \operatorname{cn}^2 sK/n}{1 - k^2\operatorname{sn}^2(sKn)\operatorname{sn}^2 u}\right\},$$

where $$\frac{\lambda}{k^2} = k^{2n-2} + \pi \operatorname{dn}^4\left(\frac{2t-1}{2n} iK'\right),$$

s is an integer from 1 to $n-1$, and t is an integer from 1 to n. (See *Proceedings of the London Mathematical Society*, Vol. XVII.). Question 8381 then follows by putting $k = 0$ and $\lambda / k^2 = 0$.

Otherwise:—The series $1 - \cos\frac{4\pi}{n} + \cos\frac{8\pi}{n} \ldots$ to n terms

$$= 1 + \cos\left(\pi + \frac{4\pi}{n}\right) + \cos\left(2\pi + \frac{8\pi}{n}\right) + \ldots \text{ to } n \text{ terms}$$

$$= 0 \text{ or } 1 \text{ as } n \text{ is odd or even}$$

$$= \cos\left(\frac{n\pi}{2} - \frac{2\pi}{n}\right)\sin(n+1)\frac{\pi}{2}\sec\frac{2\pi}{n}$$

$$= 0 + 1 - 1 + 1 - 1 \ldots \text{ to } n \text{ terms.}$$

Subtracting the second series from the first, we have

$$1 + 2\Sigma(-1)^s \cos^2 s\pi / 2n = 0.]$$

8448. When n is odd, we have

$$(-)^{\frac{1}{2}(n-1)}\sec n\theta = \frac{\sec^n\theta}{n\sec^{n-1}\theta - \frac{n(n^2-1)}{3!}\sec^{n-3}\theta + \frac{n(n^2-1)(n^2-3^2)}{5!} \times \sec^{n-5}\theta - \&c.},$$

and $\sec n\theta$ is unaltered if we put for θ any one of the $n-1$ values

$$\theta \pm \frac{2\pi}{n}, \quad \theta \pm \frac{4\pi}{n}, \ldots\ldots \theta \pm \frac{(n-1)\pi}{n}.$$

Hence $$n(-)^{\frac{1}{2}(n-1)}\sec n\theta = \sec\theta + \left\{\sec\left(\theta + \frac{2\pi}{n}\right) + \sec\left(\theta - \frac{2\pi}{n}\right) + \ldots \right.$$
$$\left. \ldots + \sec\left(\theta + \frac{n-1}{n}\pi\right) + \sec\left(\theta - \frac{n-1}{n}\pi\right)\right\}.$$

Now, $$\sec\left(\theta + \frac{n-1}{n}\pi\right) = -\sec\left(\theta - \frac{\pi}{n}\right),$$

$$\sec\left(\theta - \frac{n-1}{n}\pi\right) = -\sec\left(\theta + \frac{\pi}{n}\right) \ldots\ldots \sec\left(\theta + \frac{\frac{1}{2}(n+3)}{n}\pi\right)$$
$$= -\sec\left(\theta - \frac{\frac{1}{2}(n-3)}{n}\pi\right),$$

$$\sec\left(\theta - \frac{\frac{1}{2}(n+3)}{n}\pi\right) = -\sec\left(\theta + \frac{\frac{1}{2}(n-3)}{n}\pi\right),$$

and if $\frac{1}{2}(n-1)$ be an even number, or n of the form $4r+1$, the series within the brackets will not have a middle term, and may be written

$$-\left\{\sec\left(\theta + \frac{\pi}{n}\right) + \sec\left(\theta - \frac{\pi}{n}\right) - \sec\left(\theta + \frac{2\pi}{n}\right) - \sec\left(\theta - \frac{2\pi}{n}\right) + \ldots\ldots\right.$$

$$-\sec\left(\theta+\tfrac{1}{2}\frac{(n-3)}{n}\pi\right)-\sec\left(\theta-\tfrac{1}{2}\frac{(n-3)}{n}\pi\right)+\sec\left(\theta+\tfrac{1}{2}\frac{(n-1)}{n}\pi\right)$$

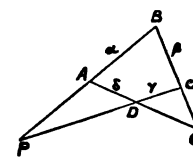
$$+\sec\left(\theta-\tfrac{1}{2}\frac{(n-1)}{n}\pi\right)\Big\}.$$

Hence, *if n be of the form* $4r+1$,

$$n\sec n\theta = \sec\theta + 2\cos\theta\,\Sigma_1^{\frac{1}{2}(n-1)}(-)^s\cos\frac{s\pi}{n} + \cos\left(\theta+\frac{s\pi}{n}\right)\cos\left(\theta-\frac{s\pi}{n}\right).$$

8480. (By Professor Genese, M.A.)—ABCD is a quadrilateral, P, Q the intersections of opposite sides. Prove that the bisectors of opposite angles A and C, B and D, P and Q meet in points which lie in threes on six straight lines; prove also that, if A, B, C, D be concyclic, the bisectors of angles P and Q meet on the join of the mid-points of AC, BD, and divide the join in the ratio AC : BD.

Solution by W. J. Johnston, M.A.; Belle Easton; *and others.*



Let α, β, γ, δ mean perpendiculars on AB, BC, CD, DA. Then the line $\alpha+\delta-\beta-\gamma$ passes through the points $(\alpha+\delta, \beta+\gamma)$, $(\alpha-\beta, \gamma-\delta)$, $(\alpha-\gamma, \beta-\delta)$, *i.e.*, through the intersections of external bisectors of A and C, of internal of B and D, and internal of P and Q.

Again, if ABCD be concyclic, suppose particles whose masses are respectively λPD, λPA, λPD, λPA to be placed at A, B, C, D. Then centre of gravity of the masses at A and C is mid-point of AC, and centre of gravity of the masses at B and D is mid-point of BD, and therefore centre of gravity of all four masses is on the join of these mid-points and divides it in the ratio PA : PD, or of AC : BD.

Again, the centre of gravity of the masses at A and B lies on AB at a distance from $P = \dfrac{PD\,.\,PA + PA\,.\,PB}{PA+PD}$, and the centre of gravity of the masses at C and D will lie on CD at the same distance from P, since PA . PB = PC . PD.

Therefore the centre of gravity of all four (which is the point previously found) lies on the internal bisector of the angle P.

8495. (By Rev. T. C. Simmons, M.A.)—Four passengers, A, B, C, D, entire strangers to each other, are travelling in a railway train, which contains l first-class, m second-class, and n third-class compartments. A and B are gentlemen whose respective *à priori* chances of travelling first, second, or third class are represented in each instance by λ, μ, ν; C and D are ladies whose similar *à priori* chances are each represented by l, m, n. Prove, for all values of λ, μ, ν (except in the particular case when $\lambda : \mu : \nu = l : m : n$), that A and B are more likely to be found both in the company of the same lady than each with a different one.

Solutions by (1) Professor IGNACIO BEYENS; (2) *the* PROPOSER.

1. La probabilité d'entrer un *gentleman* en 1e, 2e ou 3e est

$$\frac{\lambda}{\Sigma}, \quad \frac{\mu}{\Sigma}, \quad \frac{\lambda}{\Sigma}, \text{ avec } \Sigma \equiv \lambda+\mu+\nu,$$

et un *gentleman* dans une certaine voiture de 1e, 2e ou 3e classe est

$$\frac{\lambda}{l\Sigma}, \quad \frac{\mu}{m\Sigma}, \quad \frac{\lambda}{n\Sigma}$$

et une *lady* dans une certaine voiture de 1e, 2e ou 3e classe

$$\frac{1}{S}, \quad \frac{1}{S}, \quad \frac{1}{S} \ldots\ldots (1), \text{ avec } S \equiv l+m+n,$$

et un *gentleman* et une *lady* dans la même voiture de 1e, 2e ou 3e

$$\frac{\lambda}{l\Sigma S}, \quad \frac{\mu}{m\Sigma S}, \quad \frac{\nu}{n\Sigma S},$$

et dans une quelconque $\quad l.\frac{\lambda}{l\Sigma S} + m.\frac{\mu}{m\Sigma S} + n.\frac{\nu}{n\Sigma S}, \quad i.e. \ \frac{1}{S}$;

ce qui d'abord est vrai, car quelconque qui soit la voiture dont un *gentleman* soit, la probabilité de se rencontrer avec une *lady* est $1/S$ d'après la (1).

La probabilité pour se placer les quatre voyageurs formant des paires sera $\quad 1/S^2, \ 2/S^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$;

et celle de qui se rencontrent dans certaine voiture de 1e, 2e ou 3e classe, deux *gentlemen* et une *lady*, sera

$$\frac{\lambda^2}{l^2\Sigma^2}\cdot\frac{1}{S}, \quad \frac{\mu^2}{m^2\Sigma^2}\cdot\frac{1}{S}, \quad \frac{\nu^2}{n^2\Sigma^2}\cdot\frac{1}{S},$$

et dans une voiture quelconque

$$\frac{1}{\Sigma^2 S}\left(\frac{\lambda^2}{l}+\frac{\mu^2}{m}+\frac{\nu^2}{n}\right) = \frac{1}{S^2}\cdot\frac{(l+m+n)(mn\lambda^2+ln\mu^2+lm\nu^2)}{lmn\,\Sigma^2}$$

$$= \frac{1}{S^2}\left\{1+\frac{n(m\lambda-l\mu)^2+m(n\lambda-l\nu)^2+l(\mu n-\nu m)^2}{lmn\,\Sigma^2}\right\},$$

quantité toujours plus grande que $1/S^2$, et seulement lui sera égal dans le cas qui se vérifie $\quad m\lambda-l\mu=0, \quad n\lambda-l\nu=0, \quad \mu n-\nu m=0,$

c'est à dire $\quad \frac{m}{\mu}=\frac{l}{\lambda}=\frac{n}{\nu}$ ou $l:m:n::\lambda:\mu:\nu,$

et par suite la question est résolue.

2. *Otherwise* :—The chance of A and C being together $= 1/(l+m+n)$, and similarly for B and D. Hence the chance that A is with one of the ladies, and B with the other, is represented by $2/(l+m+n)^2$. Again, the chance of A and B being together is

$$\frac{\lambda^2}{l(\lambda+\mu+\nu)^2}+\frac{\mu^2}{m(\lambda+\mu+\nu)^2}+\frac{\nu^2}{n(\lambda+\mu+\nu)^2},$$

where the separate terms denote the respective probabilities that the common compartment is first, second, or third class. Hence the chance that A, B, and C are all together is $\frac{1}{l+m+n}\cdot\frac{\lambda^2mn+\mu^2nl+\nu^2lm}{lmn(\lambda+\mu+\nu)^2}$, the double of which denotes the chance that A and B are both with some *one* of the ladies.

Now $\dfrac{2}{l+m+n} \cdot \dfrac{\lambda^2mn+\mu^2nl+\nu^2lm}{lmn\,(\lambda+\mu+\nu)^2} > \dfrac{2}{(l+m+n)^2}$,

if $(\lambda^2mn+\mu^2nl+\nu^2lm)(l+m+n) > lmn\,(\lambda+\mu+\nu)^2$,

i.e., if $l(\mu^2n^2+\nu^2m^2)+m(\nu^2l^2+\lambda^2n^2)+n(\lambda^2m^2+\mu^2l^2) > 2lmn\,(\mu\nu+\nu\lambda+\lambda\mu)$,

which is evidently the case, except when $l : m : n = \lambda : \mu : \nu$.

[The Question is intended as an illustration of the principle set forth at the top of p. 74 of Vol. XLIV., and further contained in Questions 8320, 8330. It will be seen from above that if p denote the chance that A travels with a lady, and q the corresponding chance for B, pq will denote the chance, *not that A and B both travel with the same lady, but the chance that A travels with one, and B with the other.* It is, moreover, evident that the former chance is denoted by the product of p and the chance of B travelling with a lady *already known to be in A's company;* the accession of knowledge serving to modify our estimate of the particular *kind* of carriage in which she is most likely to be situated. Mr. BIDDLE remarks that—"If p denote the chance of A travelling with one or other of two ladies whose chances of travelling with him are equal, $\frac{1}{2}p$ will denote his chance of travelling with one in particular. So with B, $\frac{1}{2}q$ will be the corresponding chance. Therefore $2(\frac{1}{2}pq)$ = the chance of A travelling with one and B with the other."]

8282. (By the EDITOR.)—If a coin be dropped at random over a grating composed of parallel equidistant wires in a horizontal plane, show that, according as the distance between the wires is equal to (1) the radius, (2) the diameter, (3) the circumference of the coin, the respective probabilities that the coin will not go through without striking, are

$$p_1 = \tfrac{1}{6}\pi + \tfrac{1}{4}\sqrt{3} = \cdot 9566,\ p_2 = \tfrac{1}{4}\pi = \cdot 7854,\ p_3 = \tfrac{1}{4} = \cdot 25,$$

so that the odds are, in (1) 22 to 1, and in (2) 11 to 3, in favour of striking, and in (3) 3 to 1 against striking.

Solution by D. BIDDLE.

If we regard the position of the circumference of the descending coin to be that of a great circle on an imaginary sphere, and if we imagine this sphere to be cut by a plane rising vertically from one of the wires of the grating, we can arrive at the proportion of oblique postures which, for any given position of the centre, will prevent the coin from passing through without touching. For it is easy to see that the number of great circles which would be obstructed by the wire, supposing each to fall freely, is proportional to the plane angle which from the centre subtends the segment of the sphere above referred to.

In (1), the position of the centre varies in its horizontal distance from a wire, from 0 to $\frac{1}{2}r$, thus we have

$$p_1 = 1 - \frac{2}{\pi}\int_0^{\frac{1}{2}} \sin^{-1} x\,dx = \frac{5}{6} + \frac{2-\sqrt{3}}{\pi} = \cdot 9186.$$

In (2), the position of the centre varies from 0 to r, therefore

$$p_2 = 1 - \frac{2}{\pi}\int_0^1 \sin^{-1} x\,dx = \frac{2}{\pi} = \cdot 6366.$$

In (3), the position of the centre varies from 0 to π, hence we have

$$p_3 = 1 - \frac{1}{\pi}\left\{(\pi - 1) + \left(1 - \frac{2}{\pi}\right)\right\} = \frac{2}{\pi^2} = \cdot 2026,$$

so that, by this solution, the odds are in (1) 45 to 4, and in (2) 7 to 4, in favour of striking, whilst in (3) they are 59 to 15 against striking.

[It was desired to obtain a *general* solution of the problem, to corroborate or correct the general solution given by the PROPOSER on pp. 105—108 of Vol. I. (2nd ed.) Mr. BIDDLE's particular results, it will be observed, differ from the PROPOSER's, which have, however, been otherwise obtained, by methods applied only to the special cases (see Vol. XXXIII., pp. 48—50), by the late Mr. MONRO and Professor SEITZ. Moreover, Mr. SIMMONS, after a careful study of the solutions of Messrs. MONRO and SEITZ, states his conviction that the results arrived at by them are perfectly correct. At the same time, Mr. BIDDLE would be glad if any contributor would point out wherein lies the fallacy (if any) in his solution.

The five facts on which Mr. BIDDLE considers his solution to depend are as follows :— (1) The horizontal distance of the centre of the coin from a wire varies from zero to half the distance between two wires; (2) all the angles at which the plane of the coin may be inclined to the vertical plane passing through its centre, and parallel to the wire, are equally probable, and vary from 0 to $\frac{1}{2}\pi$; (3) the extremity of that radius of the coin which is perpendicular to the diameter lying in the vertical plane above referred to, is invariably the point, in the coin, farthest from that plane, and therefore most liable to overhang the wire; (4) the vertical plane referred to may cut the coin in any diameter indifferently; and (5), that diameter may be inclined to the horizon at any angle indifferently, that is, without affecting the result.

Mr. BIDDLE adds further, but tentatively — "that the postures or attitudes of bodies which are moving should be regarded in a somewhat different light from those of stationary bodies. In the case of a stationary disc, it is no doubt perfectly right to consider its normal as pointing with equal probability to any part of the celestial sphere, and the number of directions in space between zenith distances θ and $\theta + d\theta$, and between azimuths ϕ and $\phi + d\phi$, as measured by $\sin\theta \,.\, d\theta \,.\, d\phi$, that is, by the amount of spherical surface enclosed. But, in the case of a falling disc, what was a point becomes a line, and what was a line becomes a plane. The several possible postures should be referred primarily to the vertical line passing through the centre of the disc. In this line, the line of descent, we must regard the centre of gravity as remaining, whatever other change of relation the disc may undergo. Angular measurement alone is applicable to denote the extent of any change in the posture of the disc, and the number of positions between θ and $\theta + d\theta$ and between ϕ and $\phi + d\phi$ is measured by $d\theta \,.\, d\phi$, not by $\sin\theta \,.\, d\theta \,.\, d\phi$. For, if we imagine the normal to the disc to be the generating line of a variable cone-like figure of which the vertical line of descent is the axis, and if we suppose the disc to occupy in turn every possible posture at a uniform rate of change, revolution would (on the hypothesis that $\sin\theta \,.\, d\theta \,.\, d\phi$ is correct) become slower and slower the greater the angle between the normal and the axis, —a condition which does not follow from any law of Nature. If, however, this afterthought hold good, the several chances of striking, in the question, must necessarily be greater than either series of results above given."]

8435. (By Professor CATALAN.)—Démontrer la formule

$$\left\{\mathrm{L}\cdot\frac{-x+z+(1-2xz+z^2)^{\frac{1}{2}}}{1+x}\right\}\left\{\mathrm{L}\cdot\frac{-x+z+(1-2xz+z^2)^{\frac{1}{2}}}{1-x}\right\}=2\sum_0^\infty \mathrm{A}_n\frac{z^{n+1}}{n+1}$$

dans laquelle $\mathrm{A}_0=\frac{1}{2}\mathrm{L}\cdot\frac{1-x}{1+x}$, $\mathrm{A}_1=1+\mathrm{A}_0x$; et, à partir de $n=1$,

$$(n+1)\,\mathrm{A}_{n+1}-(2n+1)\,x\mathrm{A}_n+n\mathrm{A}_{n-1}=0.$$

Solution by D. EDWARDES.

Let u denote the expression in the question. We easily get

$$\frac{du}{dz}=2\,\frac{\log\{z-x+(1-2xz+z^2)^{\frac{1}{2}}\}}{(1-2xz+z^2)^{\frac{1}{2}}}-\frac{\log(1-z^2)}{(1-2xz+z^2)^{\frac{1}{2}}}.$$

Let $$\frac{\log\{z-x+(1-2xz+z^2)^{\frac{1}{2}}\}}{(1-2xz+z^2)^{\frac{1}{2}}}=\mathrm{C}_0+\mathrm{C}_1z+\mathrm{C}_2z^2+\&c.,$$

whence, by differentiating, we obtain

$$1-(z-x)\,(\mathrm{C}_0+\mathrm{C}_1z+\mathrm{C}_2z^2+\ldots)=(1-2xz+z^2)\,(\mathrm{C}_1+2\mathrm{C}_2z+3\mathrm{C}_3z^2+\ldots).$$

Hence we have $\mathrm{C}_0=\log(1-x)$, $\mathrm{C}_1=1+x\mathrm{C}_0$,

and $$(n+1)\,\mathrm{C}_{n+1}-(2n+1)\,x\mathrm{C}_n+n\mathrm{C}_{n-1}=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

From the first equation, and assuming

$$u=2\left(\mathrm{A}_0z+\mathrm{A}_1\frac{z^2}{2}+\mathrm{A}_2\frac{z^3}{3}+\&c.,\right)$$

we have, if the P's are LEGENDRE's Coefficients,

$$-\mathrm{A}_0+\mathrm{A}_1z+\mathrm{A}_2z^2+\&c.$$
$$=\mathrm{C}_0+\mathrm{C}_1z+\mathrm{C}_2z^2+\&c.-\tfrac{1}{2}\log(1-x^2)\,(\mathrm{P}_0+\mathrm{P}_1z+\mathrm{P}_2z^2+\ldots).$$

From this, by equating coefficients of powers of z,

$$\mathrm{A}_0=\tfrac{1}{2}\log\frac{1-x}{1+x},\quad \mathrm{A}_1=\mathrm{C}_1-\tfrac{1}{2}x\log(1-x^2)=1+x\,\mathrm{A}_0 \text{ (from above)}.$$

Also, we have $$\mathrm{A}_n=\mathrm{C}_n-\tfrac{1}{2}\log(1-x^2)\,\mathrm{P}_n,$$

and therefore $(n+1)\,\mathrm{A}_{n+1}-(2n+1)\,x\mathrm{A}_n+n\mathrm{A}_{n-1}=0$, by (1), and by observing that (1) is also the equation between three consecutive LEGENDRE's Coefficients. The actual value of A_n is $(n>1)$

$$\mathrm{A}_n=\mathrm{P}_0\mathrm{P}_{n-1}+\frac{\mathrm{P}_1\mathrm{P}_{n-2}}{2}+\frac{\mathrm{P}_2\mathrm{P}_{n-3}}{3}+\ldots+\frac{\mathrm{P}_{n-1}\mathrm{P}_0}{n}+\tfrac{1}{2}\log\frac{1-x}{1+x}\,\mathrm{P}_n.$$

8485. (By Capt. H. BROCARD.)—On sait que la droite GZ, qui joint le centre de gravité au centre du cercle de Brocard, rencontre le cercle circonscrit au triangle en un point N situé également sur les lignes OD, KZ′ et HD′, et dont les coordonnées sont proportionnelles à $1/\cos(\mathrm{A}+\theta)$, $1/\cos(\mathrm{B}+\theta)$, $1/(\mathrm{C}+\theta)$. Démontrer que le second point d'intersection V est sur la polaire du point G par rapport au cercle de Brocard, ou, en d'autres termes, que cette polaire passe par le point R diamétralement opposé à N sur le cercle circonscrit. Déterminer enfin les coordonnées du point V.

Solution par Professor G. de LONGCHAMPS.

On sait que, si une conique est représentée (coordonnées barycentriques) par $f(\alpha, \beta, \gamma) = 0$, l'equation de la polaire du point $\alpha_0, \beta_0, \gamma_0$ est

$$\alpha_0 f'_\alpha + \beta_0 f'_\beta + \gamma_0 f'_\gamma = 0;$$

par suite celle du centre de gravité est $f'_\alpha + f'_\beta + f'_\gamma = 0$. En appliquant cette relation au cercle de Brocard qui est représenté par

$$b^2c^2\alpha^2 + a^2c^2\beta^2 + a^2b^2\gamma^2 - a^4\beta\gamma - b^4\gamma\alpha - c^4\alpha\beta = 0,$$

on a

$$\alpha(b^2-c^2)^2 + \beta(c^2-a^2)^2 + \gamma(a^2-b^2)^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

Telle est l'équation de la polaire de G, par rapport au cercle de Brocard; cette forme met en évidence ce fait que cette droite passe par R, *point de Steiner*. En effet, les coordonnées de ce point sont (*Journal de Mathématiques Spéciales*, Mars 1886)

$$\frac{1}{b^2-c^2}, \quad \frac{1}{c^2-a^2}, \quad \frac{1}{a^2-b^2};$$

et l'équation (1) est visiblement vérifiée par ces valeurs. Ce point R est diamétralement opposé à N (point de Tarry), *la polaire de G par rapport au cercle de Brocard est donc la perpendiculaire abaissée du point de Steiner sur la droite qui joint le centre du cercle de Brocard ou encore le centre de gravité au point de Tarry*. Au fond, cet énoncé constitue la question proposée.

M. BROCARD demande en outre de calculer les coordonnées du point V, second point d'intersection de la polaire considérée avec le cercle circonscrit.

Il faut alors considérer les deux équations simultanées :

$$a^2\beta\gamma + b^2\alpha\gamma + c^2\alpha\beta = 0, \quad (b^2-c^2)^2\alpha + (c^2-a^2)^2\beta + (a^2-b^2)^2\gamma = 0.$$

Eliminons α, et nous avons entre β et γ une relation quadratique homogène qui, après un calcul fort simple, se décompose en deux facteurs. Nous trouvons ainsi $[\beta(c^2-a^2) - \gamma(a^2-b^2)][\beta c^2(c^2-a^2) - \gamma b^2(a^2-b^2)] = 0$. Au premier facteur correspond le point de Steiner, déjà signalé; à l'autre, correspond le point V en question; et, en désignant par $\alpha_0, \beta_0, \gamma_0$ les coordonnées de ce point, nous avons finalement

$$\frac{\alpha_0(b^2-c^2)}{a^2} = \frac{\beta_0(c^2-a^2)}{b^2} = \frac{\gamma_0(a^2-b^2)}{c^2}.$$

De ces considérations on peut tirer plusieurs conséquences intéressantes et notamment, comme je l'ai montré, une construction relativement simple du centre de la conique de Kiepert.

[Capt. BROCARD remarks that—"La question se résout facilement par l'analyse et donne un intéressant exercice de calcul. Le second énoncé m'a été communiqué depuis longtemps par M. M'CAY. Les cercles GA"B", GB"C", GA"C" sont ceux qui portent actuellement le nom de cercles de M'CAY (voir CASEY, p. 253—255.)

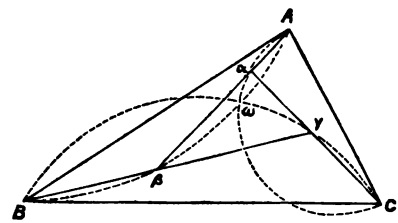
Professor LONGCHAMPS adds the following references:— *Journal de Mathématiques Spéciales* (No. d'Avril 1886). Sur la Conique de KIEPERT voyez; *Journal de Mathématiques Spéciales* – (1) *Propriétés de l'Hyperbole des neuf points*, par M. BROCARD (1885, p. 12); et (2) *Sur le point de Steiner*, par M. NEUBERG (1886, p. 6)].

8271. (By CHARLOTTE A. SCOTT, D.Sc.)—ABC is a triangle; lines $A\alpha\beta$, $B\beta\gamma$, $C\gamma\alpha$ are drawn, making equal angles respectively with AB, BC, CA. Show that (1) all the triangles so formed have one Brocard-point fixed,

while the other describes a circle concentric with ABC; and (2) any point connected with the triangle describes a circle, while any straight line passes through a point.

Solution by the Rev. T. C. SIMMONS, M.A.

1. On BC, CA, AB describe three circular arcs, touching respectively CA at C, AB at A, and BC at B. They will intersect at the Brocard-point ω, which, if the triangle is *positively* lettered (*i.e.*, if, as in the figure, in moving from A to C round the angle ABC, we revolve in the *positive* sense), will be the *positive* B-point of the triangle. Let θ be the B-angle, and λ the angle which the sides of the new triangle make with the sides of ABC. Then, if on the three arcs we take $C\gamma : C\omega = A\alpha : A\omega = B\beta : B\omega = \lambda : \theta$,



it is clear that $A\alpha$ makes an angle λ with AB, and likewise subtends an angle λ at C; and similarly for $B\beta$, $C\gamma$. Thus $A\alpha\beta$, $B\beta\gamma$, $C\gamma\alpha$ will be three straight lines, and $\alpha\beta\gamma$ will be the triangle under consideration.

Moreover, $\angle\, \omega\beta\gamma = \omega B\gamma + B\omega\beta = \omega B\gamma + BA\beta = \omega B\gamma + CB\beta = \theta$,

and similarly angles $\omega\gamma\alpha$, $\omega\alpha\beta$ each $= \theta$, showing that ω is the positive B-point of the triangle $\alpha\beta\gamma$.

2. Hence, since $\alpha\beta\gamma$ is a figure of invariable form, though varying magnitude, revolving round ω as a fixed centre, it follows (see TOWNSEND'S *Modern Geometry*, Vol. I., p. 35), that all points invariably connected with it describe, and all lines invariably connected with it envelop, similar figures. Now, the points α, β, γ are known to describe circles, and the lines $\alpha\beta$, $\beta\gamma$, $\gamma\alpha$ to envelop points. Hence, *all* points connected with $\alpha\beta\gamma$ describe circles, and *all* lines connected with $\alpha\beta\gamma$ envelop, that is to say, pass through, fixed points.

To find the particular circle described by the negative B-point ω', we notice that, when $\lambda = 0$, ω' coincides with the negative B-point of ABC, and when $\lambda = \theta$, it coincides with ω the positive B-point, both of which, are at the same distance from O the circum-centre. Again, when the sides of $\alpha\beta\gamma$ are perpendicular to ωA, ωB, ωC, the circum-circle of ABC is the auxiliary circle of the B-ellipse (called in Quest. 8375 the *sine-circle*) of $\alpha\beta\gamma$. Hence, in this case also, ω' is at the same distance from O as ω. And as three points on the circular locus of ω' are thus proved to be equidistant from O, it follows that O is the centre of the circle described by ω'

8481. (By the EDITOR.)—Through a given point O in the side AB of a triangle ABC draw a straight line POQ, cutting the side AC in P, and the prolongation of the side CB in Q, so that PC may have to QB the given ratio of $m : n$.

Solutions by (1) D. BIDDLE; (2) R. F. DAVIS, M.A.

1. Draw OK parallel to AC, and make KL such that

$$OK : KL = PC : BQ,$$

[*i.e.*, = the *given* ratio $m : n$]. With centre B and radius BL, draw the arc LM cutting the semicircle on BC, in M, and draw MN perpendicular to BC. Make NR = 4KL, and draw RS at right angles to BC, to cut the semicircle in S. With centre B and radius BS describe the arc ST cutting BC in T; and, finally, make BQ = ½LT. The line drawn from Q through O is the line required.

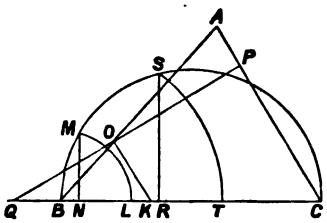


FIG. 1.

For $$BQ = \tfrac{1}{2}LT = \tfrac{1}{2}(BT - BL) = \tfrac{1}{2}\{(BR)^2 - BL\}$$

$$= \tfrac{1}{2}\{(BN + 4KL)^2 - BL\} = \tfrac{1}{2}\{[(BK - KL)^2 + 4KL]^{\frac{1}{2}} - (BK - KL)\},$$

which is the value of BQ obtained as follows:—

$$OK : KL = PC : BQ;\ PC : OK = CQ : KQ = BC + BQ : BK + BQ;$$

$$\therefore\ BQ : KL = BC + BQ : BK + BQ,\ BQ^2 + (BK - KL)\,BQ = BC \cdot KL,$$

whence, taking BC = unity, BQ = &c.

Where PC < BQ, it is advisable to find PC, by reference to the two equations, QB = *m*PC, and QB + BK : OK = QB + BC : PC, whence

$$PC = \tfrac{1}{2}\{OK - (n/m)\,BK \pm [(OK - (n:m)\,BK)^2 + 4\,(n/m)\,BC \cdot OK]^{\frac{1}{2}}\},$$

which can be rendered geometrically in the same way.

[A simpler construction and proof may be obtained as follows:—

If, in Mr. BIDDLE's Fig. 1, we produce KB to Q, so that LQ . QB = BC . KL (a simple problem), then POQ will be the line required. For the construction gives BC : QB = QL : LK, whence CQ : QB = QK : LK; but PC : CQ = OK : QK; and from the last two proportions it follows at once that PC : QB = OK : KL = $m : n$.]

2. *Otherwise*:—On CA take a point A′ so that CA′ : BC = $m : n$, and draw a parabola touching CA′, CB at the points A′, B. Then any tangent to this parabola will intersect CA′, CB in points P, Q such that

$$CP : CA' = BQ : BC,$$

or

$$CP : BQ = CA' : BC = m : n.$$

If then O be any given point (not necessarily in AB), the two tangents from O to the parabola will be the required positions of the transversal. If the parabola cut AB in β, the problem is impossible when O lies on Bβ. If AC be produced to A″ so that CA″ = CA; then, proceeding as before, we get construction for case in which CP : BQ is negative.

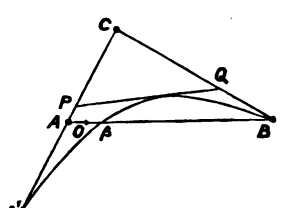


FIG. 2.

8479. (By Professor HUDSON, M.A.)—If two straight lines A, B meet a quartic each in four points A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4, and two parabolas be described through the points A_1, A_2, B_1, B_2, A_3, A_4, B_3, B_4; prove that the remaining points of intersection of the parabolas and the quartic lie on a conic section.

Solution by W. J. C. SHARP, M.A.; W. J. BARTON, M.A.; *and others.*

For parabolas, conics may be read throughout this question. The two conics through the given points form a quartic system through sixteen points on the quartic, eight of which are the given points and eight the additional intersections of the conics with the quartic. Now, the two lines A and B and any conic through five of the additional points form a quartic system through thirteen ($4^2 - \frac{1}{2} \times 3 \times 2$) of the points of intersection, and therefore through all (SALMON'S *Higher Plane Curves*, Art. 33.)

[The theorem may be generalized into the following one:—If the eight intersections of a conic and a quartic be taken four and four, and through each set of four points a conic be described, the other eight intersections of these conics with the quartic lie on a fourth conic. By supposing the lines A and B to coincide, and taking the lines A_1B_1, A_2B_2; A_3B_3, A_4B_4 as the conics, it appears that the eight points tangential to four points in one straight line lie on a satellite conic.

It may also be deduced that, if the points of contact of a bitangent be joined two and two, a conic can be described touching the quartic at the remaining intersections of those lines; and that, if points of contact of two inflexional tangents be joined, and also their tangentials, the remaining intersections of these lines are such that a conic can be described osculating the quartic at the two points on the chord through the inflexions and passing through the remaining intersections.]

8471. (By Professor NEUBERG.)—Trouver dans le plan d'un triangle ABC un point M tel que les perpendiculaires menées sur MA, MB, MC respectivement de A, B, C, ou de B, C, A, ou de C, A, B, concourent en un même point N ou N_1 ou N_2. Démontrer que les triangles ABC, NN_1N_2 sont équivalents et ont même centre de gravité.

Solution by R. F. DAVIS, M.A.; R. KNOWLES, B.A.; *and others.*

The conditions of the question require (1) M, N to be the extremities of a diameter of the circum-circle; (2) and also (AN, BN_1, CN_2), (BN, CN_1, AN_2), (CN, AN_1, BN_2) to be parallel. If *abc*, nn_1n_2 are two equilateral triangles inscribed in the same circle, the conditions (2) are satisfied. Projecting orthogonally, we get the following construction. About the given triangle ABC describe an ellipse having its centre at the centroid G and its tangents at the angular points parallel to the respectively opposite side. The osculating circles at A, B, C will then cointersect in a point N, which is the fourth point of intersection of the ellipse and the circum-circle. This determines N; then M is the extremity of the diameter of the circum-circle through N, and N_1, N_2 are points on the ellipse whose

excentric angles differ from that of N by $2\pi/3$. The triangles ABC, NN_1N_2, being the projections of equilateral triangles inscribed in the same circle, have consequently the same centroid and area.

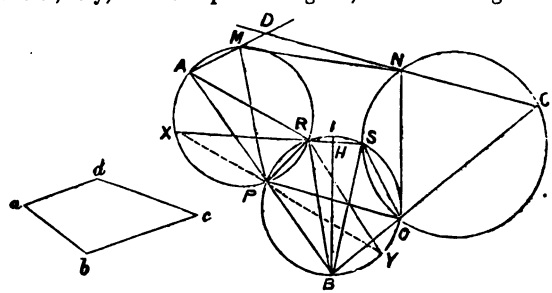
There does not seem to be any simple geometrical construction for the point N. The trilinear equations of the ellipse and circum-circle being respectively $(a\alpha)^{-1}+\ldots+\ldots=0$ and $a/\alpha+\ldots+\ldots=0$, the coordinates of N are given by the proportion $a\alpha : b\beta : c\gamma = (b^2-c^2)^{-1} : \ldots : \ldots$. N may therefore be regarded as the centre of gravity of three masses inversely proportional to b^2-c^2, c^2-a^2, a^2-b^2, placed at the angular points. Hence divide AC (the mean side) in L so that $AL : CL = b^2-c^2 : a^2-b^2$; then BL produced will meet the circum-circle in N. The trilinear equation to GN is found to be $a\alpha(b^2-c^2)(b^2+c^2-2a^2)+\ldots+\ldots=0$.

[The Proposer remarks that—"Le point M de cette question est le *point de* Tarry, N est le *point de* Steiner, N_1 et N_2 sont les *conjugués cyclotomiques des points de Brocard*. Ces points N_1, N sont respectivement à l'intersection des segments capables des angles A, B, C construits vers l'intérieur du triangle ABC sur les côtés AB, BC, CA ou sur AC, CB, BA. Pour les références, voir la Question 8445, et *Mathesis*, t. v., p. 208; t. vi., p. 5.]

8425. (By Arthur Hill Curtis, LL.D., D.Sc. Suggested by Quest. 8354.)—Inscribe in a given quadrilateral a quadrilateral similar to a given one.

Solution by Professor Edmund Bordage.

Let us suppose, on the contrary, that we have to circumscribe to the smaller quadrilateral a quadrilateral similar to the given one. Let MNOP be the smaller quadrilateral, and ABCD the quadrilateral similar to *abcd* that we have to circumscribe to the quadrilateral MNOP. The points A, B, and C are lying upon three segments capable of the angles A, B, and C; nay, AB must pass through P, and BC through O.



If we could determine the intersection point B of these lines upon the arc PBO, the question would be resolved. Let APB be the direction of the side AB; if the straight lines AR and BR are drawn, we shall obtain a triangle similar to all the triangles XRY, having the same summit R,

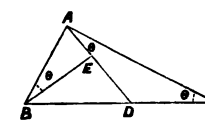
and a secant such like XY passing through P as basis, for the angles A and X, B and Y are equal, because they have the same measure. The ratio BR / BA is then constant, and equal to the ratio of the radius r of the circle PRO with the distance d of the centres of PBO and MAP.

We shall then have $\frac{BR}{BA} = \frac{p}{d}$, and by a similar reasoning, d' being the distance between the centres of PBO and OCN, $\frac{BS}{BC} = \frac{p}{d'}$; the division gives us $\frac{BR}{BS} \cdot \frac{BC}{BA} = \frac{d'}{d}$, and since $\frac{BC}{BA} = \frac{bc}{ba}$ the ratio $\frac{BR}{BS} = \frac{d'}{d} \cdot \frac{ba}{bc}$ is determined. But we know two points R and S of the circumference POB, and the ratio of the distances RB and SB to a third point B. This point is easily determined, by a very elementary construction of geometry.

We shall have two points for the question, and since we can successively take, as side corresponding to one of the angles a, b, c, d, every one of the four sides of MNOP, the question shall generally have *eight* solutions.

Now, it is easy to satisfy the enunciation of the direct problem. We shall take for MNOP a quadrilateral similar to the small quadrilateral, and we shall construct the circumscribed quadrilateral similar to the large quadrilateral. It will suffice to make a figure similar to this, in which the circumscribed quadrilateral shall be equal to the largest given quadrilateral.

8497. (By R. Tucker, M.A.)—In a triangle ABC, $\angle ABC = 3ACB$: $\angle CAD = ABE = ACB = \theta$; show that the ratio of triangle BED to triangle ABC is $= \sin^2 5\theta \sin\theta : \sin^2 4\theta \sin 3\theta$, and that, when $\theta = 22\frac{1}{2}°$, 30°, this reduces to $1 : 2\sqrt{2}$ and $1 : 6$ respectively.



Solution by W. W. Taylor, M.A.; R. H. W. Whapham; *and others.*

$$\triangle BED : \triangle ABC = BE \cdot BD \sin 2\theta : AB \cdot BC \sin 3\theta$$
$$= BE \cdot BD \cdot AB, \sin 2\theta : AB \cdot AB \cdot BC \sin 3\theta$$
$$= \sin 5\theta \sin 5\theta \sin\theta \sin 2\theta : \sin 4\theta \sin 2\theta \sin 4\theta \sin 3\theta,$$

whence the stated result follows at once.

Note on Question 7833. *By* Rev. T. P. Kirkman, F.R.S.

From Mr. Carr's remarks on p. 118 of Vol. 40, it would appear that Mr. Carr claims to know that no incompressible body exists. If he has not explored the cosmos, he can know that an incompressible body

does not exist only by proof that it is absurd to affirm or suppose one. This magnifies the power of his logic. He can supply a correct finite numerical answer to a question containing a condition demonstrably absurd! This criticism of mine will be thrown away on thinkers who convince themselves that nonsense, if it be only vast enough, becomes sense and science, just as rapine and murder, if vast enough, become praise and glory.

If Mr. CARR had omitted his ( ), this criticism could not stand upon a laughable example of proving too much. The only reply, if the ( ) were out, would be, Produce your function giving $\infty \times 0$ = finite, in its connection with the problem. With an assumed $0 / 0 = A$ ($= 4\,m\nu^2$ in this case) you may answer anything nearly. You confess that you are humbugging us, when you give to a question as answer that is intended to be *not nothing*, either 2 . 0 or 3 . 0, double or treble nothing; that *is* nonsense. But you maintain that the answer ∞ . 0, *i.e.*, infinite nothing, is both sense and science, and definite too. An infinite nothing is—like a square circle, a necessary cause, a moving equilibrium, an instantaneous velocity, an infinite energy of laziness, an eternal standstill of motion, an honest abstraction with refusal to restore, and basest and easiest of all, a false and foul accusation of man-shaping—a most sublime profundity of philosophy.

8383. (By A. F. THEODOSIUS.)—Prove that the radical centre of the three escribed circles of a triangle lies on the straight line joining the centre of the inscribed circle of the triangle to its centre of gravity.

Solution by R. F. DAVIS, M.A.; EMILY PERRIN; *and others.*

The radical axis of the escribed circles opposite to A, B passes through the middle point of AB and a point on BC whose distance from B $= \frac{1}{2}(a+b)$; for it bisects the common tangents. Its equation in trilinear coordinates is therefore $(a+b)(a\alpha - b\beta) + c\gamma(a-b) = 0$. The radical centre is then determined thus, $a\alpha : b\beta : c\gamma = (b+c) : (c+a) : (a+b)$, and therefore lies on the line $a\alpha(b-c) + \ldots + \ldots = 0$, which joins $(\alpha = \beta = \gamma)$ with $(a\alpha = b\beta = c\gamma)$.

8187. (By the late Professor CLIFFORD, F.R.S.)—The quantities $\xi + i\eta$, $x + iy$ being connected by the equation $x + iy = dn(\xi + i\eta)$, find (1) the relation between x and y which makes ξ or η constant; and (2) show that the curves ξ = const. cut the curves η = const. at right angles; and draw figures indicating the general form of the curves.

Solution by W. J. CURRAN SHARP, M.A.

From the given equation, we have

$$x^2 - y^2 = 1 - k^2(\xi^2 - \eta^2), \quad xy = -k^2\xi\eta;$$

and the curves are $x^2-y^2 = 1-c^2+\frac{4x^2y^2}{c^2}$, $x^2-y^2 \quad 1+c'^2-\frac{4x^2y^2}{c'^2}$,

which are easily drawn.

[The quantities ξ and η are conjugate functions, and the curves ξ = constant and η = constant cut at right angles. (CLERK MAXWELL'S *Electricity*, p. 263, Ed. 2).]

8441. (By Professor HUDSON, M.A.)—If

$$x^2 \cos\alpha \cos\beta + x(\sin\alpha + \sin\beta) + 1 = 0,$$

and

$$x^2 \cos\beta \cos\gamma + x(\sin\beta + \sin\gamma) + 1 = 0,$$

prove that

$$x^2 \cos\gamma \cos\alpha + x(\sin\gamma + \sin\alpha) + 1 = 0.$$

Solution by G. G. STORR, B.A.; A. MUKHOPÂDHYÂY, M.A.; *and others.*

Solving for $\cos\beta$ we have $x\cos\beta \sin\frac{1}{2}(\alpha+\gamma) = \cos\frac{1}{2}(\alpha+\gamma)$, and for $\sin\beta$, $x\sin\beta \sin\frac{1}{2}(\alpha+\gamma) = \sin\frac{1}{2}(\alpha+\gamma) + x\cos\frac{1}{2}(\alpha-\gamma)$.

Squaring and adding, we obtain

$$x^2[1-\cos(\alpha+\gamma)] = 2 + 2x(\sin\alpha+\sin\gamma) + x^2[1+\cos(\alpha-\gamma)],$$

or

$$x^2\cos\alpha\cos\gamma + x(\sin\alpha+\sin\gamma) + 1 = 0.$$

8496. (By W. J. C. SHARP, M.A.)—If the number of partitions, without repetitions, of a number n be denoted by P_n, and the number with repetitions by Q_n; prove that

$$P_n + P_{n-2}Q_1 + P_{n-4}Q_2 \ldots\ldots = Q_n;$$

$$P_n + P_{n-1}Q_1 + P_{n-2}Q_2 + \ldots\ldots + Q_n = P_{2n} - P_{2n-1}Q_1 + P_{2n-2}Q_2 \ldots\ldots + Q_{2n};$$

$$P_{2n-1} - P_{2n-2}Q_1 + P_{2n-3}Q_2 - \ldots\ldots - Q_{2n-1} = 0.$$

Solution by Professor LLOYD TANNER, M.A.

We have $P(x), = 1 + P_1x + P_2x^2 + \ldots\ldots, = (1+x)(1+x^2)(1+x^3)\ldots\ldots,$

$$Q(x), = 1 + Q_1x + Q_2x^2 + \ldots\ldots, = 1/(1-x)(1-x^2)(1-x^3)\ldots\ldots,$$

therefore $P(x)\,Q(x^2) = Q(x)$ and $P(-x)\,Q(x) = P(x^2)\,Q(x^2)$;

therefore $\quad co(n) \,.\, P(x)\,Q(x^2) = co(n) \,.\, Q(x)$;

$co(n) \,.\, P(x)\,Q(x), = co(2n) \,.\, P(x^2)\,Q(x^2), = co(2n)\,P(-x) \,.\, Q(x)$;

and $\quad co(2n-1) \,.\, P(-x)\,Q(x), = co(2n-1) \,.\, P(x^2)\,Q(x^2), = 0$;

where $co(m) \,.\, \phi$ means the coefficient of x^m in ϕ. These are the required results.

8437. (By Professor WOLSTENHOLME, M.A., Sc.D.)—Normals to a parabola ($y^2 = 4ax$) at P, Q, R meet in a point (X, Y); prove that the straight lines drawn from P, Q, R perpendicular to the opposite sides are normals to the parabola $y^2 = 16a(x + 12a)$, and that, if the corresponding tangents be drawn forming a triangle P'Q'R', this triangle will be equal and similar to PQR, each of the three distances PP', QQ', RR' being parallel to the axis and of length $10a + X$.

Solution by Rev. T. R. TERRY, M.A.; R. F. DAVIS, M.A.; *and others.*

If the coordinates of P are $(ap^2, 2ap)$, and so for Q and R; then p, q, r are the roots of $am^3 + m(2a - X) - Y = 0$; therefore

$$p + q + r = 0 \text{ and } pq - r^2 = (2a - X)/a \quad \ldots\ldots(1).$$

The equation of QR is $(q + r)y - 2x - 2aqr = 0$, and that of the perpendicular from P on QR is $2y - px - 4ap + ap^3 = 0$ $\ldots\ldots(2)$.

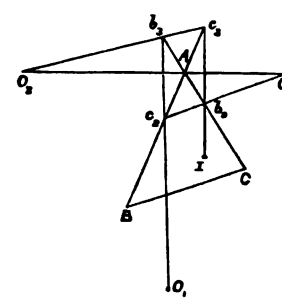
Again, the coordinates of any point on the second parabola are $(af^2 - 12a, 4af)$; and the tangent and normal are

$$fy = 2x + 2af^2 + 24a, \quad 2y + fx + 4af - af^3 = 0 \quad \ldots\ldots(3).$$

From (2) and (3), we see that (2) is normal where $f = -p$, hence tangent Q'R' is $py + 2x + 2ap^2 + 24a = 0$, which is parallel to QR; so R'P' is $qy + 2x + 2aq^2 + 24a = 0$, whence coordinates of R' are $ar^2 - 10a - X$ and $2ar$; therefore R'R is parallel to axis and $= 10a + X$.

8374. (By W. VOYSON, B.A.)—If through O_2, O_3, two of the ex-centres of a triangle ABC, lines $O_2b_2c_2$, $O_3b_3c_3$ be drawn parallel to BC, meeting AC, AB in b_2, b_3, c_2, c_3 respectively, prove that the line c_2b_2 will pass through the in-centre and c_3b_3 through the third ex-centre.

Solution by Professor NILKANTHA SARKAR, M.A.; *the* PROPOSER; *and others.*



Since $| Bc_2Ac_3 |$, $| Cb_2Ab_3 |$ are two homographic ranges on AB and AC, therefore

$$O_2 | Bc_2Ac_3 | = O_3 | Cb_2Ab_3 |,$$

and these pencils are in perspective, therefore I, b_2, c_3 are collinear. Also

$$O_3 | Bc_2Ac_3 | = O_2 | Cb_2Ab_3 |,$$

and they are in perspective, therefore O_1, c_2, b_3 are collinear.

APPENDIX I.

UNSOLVED QUESTIONS

By Professor Sylvester, F.R.S.

1453. Show that a given point is *inside*, *outside*, or *neither* inside nor outside, a given conic, according as the product of the discriminant by the point-function is *positive*, *negative*, or *zero*.

1850. In a parabolic segment cut off by a line perpendicular to the axis two points are taken; find the mean value of such of the chords drawn through them as cut the segment in two points.

1856. Prove that the Jacobian of the time of a planet's describing any arc, the chord of the arc, and the sum of the two extreme distances from the sun, in respect to the eccentricity and two extreme eccentric anomalies, is zero; and hence deduce the time for a planet or comet in terms of the said chord and sum.

1930. A uniform ellipsoid with fixed centre rolls on an indefinitely rough plane. Determine the time, pressure, and friction of the plane corresponding to any given position of the body. Find also the same when the axes of the ellipsoid remain the principal axes of the body, but the principal moments of the latter are in any given ratios to one another: state the condition in order that, during the whole continuance of the motion, the pressure and friction may be nil, and also in general under what circumstances the time will be expressible by an elliptic function.

2231. If a system of conics touch a given straight line at a given point at which they have all the same curvature, and pass through a second given point, show that, on taking as axis of coordinates the bisectors of the angle between the given line and the line joining the two given points, the locus of the foci of the conics so taken consists of the two circular cubics

$$2by(x^2+y^2)-(ax+by)^2=0, \quad 2ax(x^2+y^2)-(ax+by)^2=0,$$

where a, b are the x, y coordinates of the second given point.

2337. By the form probability of a group, understand the probability of a group of a given number of points satisfying a prescribed condition of form.

Let ABCD be any quadrilateral; ϕ the form probability of a group of points limited to the area ABCD; p, q, r, s the corresponding probabilities when one of the group is restricted to the lines AB, BC, CD, DA respectively. Show that, if four forces act along the successive sides of

ABCD, regarded as a rigid figure, with intensities proportional to $\phi-p$, $\phi-q$, $\phi-r$, $\phi-s$ respectively, they will keep the quadrilateral at rest.

2352. DEF.—*We may use* P_*Q *to denote the third point in which the right line* PQ *meets a given cubic;* P_*Q_*R *to denote the third point in which the line joining the one last named and* R *meets the cubic, and so on. Thus* P_*P *will denote the tangential or point in which the tangent at* P *meets the given cubic, and* $[P_*P]_*[P_*P]$ *will denote the second tangential, i.e., the tangential to the tangential at* P.

1. Prove that $[P_*P]_*[P_*P] = I_*P_*[P_*P]_*P_*I$, where I is any point of inflexion in the given curve.

2. Obtain a function of P, I which shall express the point in which the curve is cut by a conic having five-point contact with it at P.

2391. Let μ points be given on a cubic curve. Through them draw any curve (simple or compound) of degree ν; the remaining $3\nu-\mu$ (say μ') points may be termed a first residuum to the given ones. Through these μ' points draw any curve of degree ν'; the remaining $3\nu'-\mu'$ points may be termed a residuum of the second order to the given ones; and in this way we may form at pleasure a series of residua of the third, fourth, and of any higher order. If μ is of the form $3i-1$, a residuum of the first or any odd order; and if μ is of the form $3i+1$, a residuum of the second or any even order in such series, may be made to consist of a single point, which I call the *residual* of the original μ points. Prove that any such residual is dependent wholly and solely on the original μ points, being independent of the number, degrees, and forms of the successive auxiliary curves employed to arrive at it.

2372. 1. Admitting the fact that $9x^3+9x^2-1$ contains an infinite number of primes for positive odd values of x, prove that there exists an infinite number of primes of the form $18n-1$ resolvable into a sum of two cubes, integral or fractional.

2. Understanding by a derivative to an arbitrary point on a cubic curve a second point also on the curve, deduced from the first by any law of construction which leads to that point alone, prove by the aid of (1) that every derivative must be either the first tangential or the second tangential, or the connective of the second tangential with the given point, or the connective of this connective with the first tangential, or of this last connective with the given point, and so on *ad infinitum*, each new point being the connective of its antecedent with the given point and with the first tangential alternately.

3. Prove that (α) the coordinates of the successive terms in this scale of derivatives (including the given point itself), expressed in terms of the coordinates of the given point, are, in respect of them, of the degrees 1, 4, 16, 25, 49, 64, 100, &c.; and (β) if these successive derivatives are denoted by P_1, P_2, P_4, P_5, P_7, P_8, P_{10}, &c., the connective of P_i and P_j is P_{i+j} or P_{i-j} according as i, j are congruous or incongruous in respect to the modulus 3.

[By the connective to any two points in a cubic is to be understood a third point in which the curve is cut by the line joining the first two.]

2431. *Definition.*—Let P, P′ be a pair of points in which the cubic curve $x^3+y^3=17$ is cut by any line parallel to its asymptote; let the tangents at P, P′ respectively cut the cubic in Q, Q′; let the lines PQ′, P′Q cut the cubic in R, R′; let PR′, P′R cut the cubic in S, S′ respectively and so

on *ad infinitum*; then the point sequences PQRS....., P'Q'R'S'..... may be called the Ramification of the radical points P, P'.

Required to prove that all commensurable solutions whatever of the equation $x^3+y^3=17$ are given by the coordinates of the Ramification whose radical points are $x=\frac{1}{7}^2$, $y=-\frac{1}{7}$; $x=-\frac{1}{7}$, $y=\frac{1}{7}^2$; and hence show that $\left(\frac{104940}{40831}\right)^3+\left(\frac{11663}{40831}\right)^3$ is the simplest possible representation of 17 as the sum of two positive rational cubes.

2492. Prove that the mean distance between any two points taken arbitrarily within a triangle is a constant fraction of the mean of the mean distances of the angles from the several points in the opposite sides, and determine the value of the fraction. Show also that the same is true of the mean distance between two points in any parallelogram, and determine the value of the fraction for that case.

2531. 1. Let E denote the event of 3 points forming the apices of an acute angle triangle. Let p be the probability of E for 3 points taken anywhere inside a given triangle A; t the probability of E when the three points lie respectively anywhere upon the three sides of A; q, r, s the several probabilities of E when one point is taken at an angle of A, a second anywhere in the side opposite that angle, and a third anywhere inside A. Prove that $p=\frac{1}{8}q+\frac{1}{8}r+\frac{1}{8}s+\frac{5}{8}t$.

2. Obtain an analogous theorem for decomposing the probability of E when the original range of the three points is a tetrahedron.

3. Also give the corresponding theorems of decomposition when E denotes the event of a group of n points satisfying any intrinsic condition of form, and the original range of the group is a triangle, parallelogram, tetrahedron, or parallelopiped.

2552. Show how for any value of n to construct a tesselated square (symmetrical about each diagonal) having $2^{2n-1}+2^{n-1}$ white tiles and $2^{2n-1}-2^{n-1}$ black tiles in half the number of rows or columns, and *vice versâ* $2^{2n-1}-2^{n-1}$ white and $2^{2n-1}+2^{n-1}$ black in the remaining rows or columns. Prove also that if such square be regarded as a magnified chess-board, and the line or column from which a Castle moves (but not its place in the line or column) be given, the chances of the place it moves *to* being like or opposite in colour to the place it moves *from* will be equal.

2589. Let a tree springing from the ground at A rise to B, thence bifurcate into BC, BD, again bifurcate at C and at D, and so on for any number n of times, thus giving rise to $1+1+2+4+\ldots+2^{n-1}$, that is 2^n points of salience (A included); prove that the ramification may be constructed subject to the condition that the right line connecting any two points of salience not springing immediately one from the other shall pass through a third point of salience.

[By the points of salience are meant the joints where bifurcation takes place and the point where the tree rises from the ground.]

2572. 1. If h, k, l... are the distinct prime factors of n, prove that the number of points (which I term pluperfect points of the nth order) at which a cubic curve can have the highest degree of contact with a curve of the degree n (not composed of repetitions of a curve of lower degree) is $9n^2(1-h^{-2})(1-k^{-2})(1-l^{-2})$.

2. Show that if the first tree in the solution to Quest. 2473 (Vol. VIII., p. 106) be planted at a pluperfect point of the nth order in a cubic, the

sequence of tree-marks 1, 2, 4, 5, 7 ... may be replaced by recurring periods of $2n$ numerals, and that the two halves of each period will consist of the same n numerals arranged in reverse order, that in fact only the first n of the numerals 1, 2, 4 ... need appear in the result.

3. Hence prove that n trees may be so arranged as to contain between them $\mathrm{E}\left\{\frac{1}{6}n\,(n-1)\right\}-\mathrm{E}\left\{\frac{1}{3}n\right\}$ rows of three in a row, where E (the symbol of entirety) denotes that the integer part only is to be taken of the function which it governs.

[Thus, for 81 trees the number will be 1053 instead of 800, the number obtained when the first tree is at a non-pluperfect point; and for 15 trees the number is 30 instead of 26, the number stated in the Editorial reference at the end of the solution above cited as applicable to 15 points, but which is, in fact, the number for 14, according to the formula stated above. It must, however, be observed, that this formula does not in general give the absolute maximum. Thus, for $n = 9$ the formula gives only 9 rows, whereas the true maximum (dealing with *real* trees) is 10. In fact, $\mathrm{E}\left\{\frac{1}{6}n\,(n-1)\right\}$ is the arithmetical limit obtained by dividing the number of duads of n by 3, the number of them in each row. Thus, $\mathrm{E}\left\{\frac{1}{3}n\right\}$ is the difference between this limit and the actual number of the formula, which may also be expressed as equal to $\frac{1}{6}(n^2-3n)$ when this is an integer, and the integer immediately superior to it when it is fractional. Thus we see in a moment that with 1000 trees, the number of rows containing three in a row may be made equal to 166167. In a word, when the first tree is planted at an ordinary point in the cubic, the number of rows given by the formulæ in Quest. 2473 is $\mathrm{E}\left\{\frac{1}{6}(n-1)^2\right\}$, when at a pluperfect point the number is $\mathrm{E}\left\{\frac{1}{6}(n-1)(n-2)\right\}$.

Another method of placing points on a cubic which gives 10 rows for 9 points, and possibly in all cases the true maximum, forms the subject of a subsequent question.]

2656. Show how to arrange 3^i points as the apices of a polygon so that each of them shall be *a ganglionic centre* to the system composed of all the others, with the exception of its own immediate antecedent and consequent; understanding by a ganglionic centre to a set of points, the point (when such exists) in which right lines joining them two and two together all intersect. A system of points admitting of such a centre may be termed a ganglionic system, and thus there are to be in the case supposed 3^i distinct ganglionic systems, each containing 3^i-3 points.

2717. Find the polar equation (a) to the general involute of the logarithmic spiral, (b) to the *second* involute of the circle.

2758. 1. Prove that the curve designated by the polar equation

$$\theta = 2\left(\frac{a \pm r}{a}\right)^{\frac{1}{2}} - \tan^{-1}\left\{\frac{1}{2}-\left(\frac{a}{a \pm r}\right)^{\frac{1}{2}}\right\}$$

has for its *fourth* evolute a circle with centre at the origin and radius $= \frac{1}{2}a$.

2. Find also the polar equation to its second evolute.

2765. Find the general values of u and v, as rational integral functions of x, which satisfy the equations

$$u^2 + l\left(\frac{du}{dx}\right)^2 = v^2 \ldots\ldots(1); \quad u^2 + l\left(\frac{du}{dx}\right)^2 = v^3 + \lambda\left(\frac{dv}{dx}\right)^2 \ldots\ldots(2).$$

2784. Given an equation-system (six equations to find six quantities) as follows :—

$$x+y+z=n, \qquad \xi+\eta+\zeta=-n,$$

$$i(y+z)+j(z+x)+k(x+y) = xy+xz+yz-\xi\eta-\xi\zeta-\eta\zeta = i(\eta+\zeta)+j(\zeta+\xi)+k(\xi+\eta),$$

$$iyz+jzx+kxy = xyz-\xi\eta\zeta = i\eta\zeta+j\zeta\xi+k\xi\eta;$$

show that there are in all seven solutions which break up into two groups, one a group of three determinable by a cubic equation, the other a group of four subdivisible into two pairs depending on the solution of a double quadratic equation. Also find what happens to the two systems of solutions when two of the coefficients i, j, k become equal to one another.

2792. If $$r = \frac{(\theta^2 \pm A\theta + B)^i (\theta^2 \pm C\theta + D)^j}{(\theta^2 \pm E\theta + F)^{i+j}},$$

where i, j are any given positive or negative integers, be the polar equation to a spiral; show that there are two, and only two, systems of values of A, B, C, D, E, F for which the curve will be rectifiable, and that in one of these systems $A = C = E = 0$.

2824. If $$x+y+z=m, \quad \xi+\eta+\zeta=\mu,$$

$$i(y+z)+i(z+x)+k(x+y) = xy+xz+yz-\xi\eta-\xi\zeta-\eta\zeta = i(\eta+\zeta)+j'(\zeta+\xi)+k'(\xi+\eta),$$

$$iyz+izx+kxy = xyz-\xi\eta\zeta = i\eta\zeta+j'\zeta\xi+k'\xi\eta,$$

where $j'+k'=i+k$; prove that the system is of the eighth order (*i.e.*, admits of eight solutions), and that the values of x, y, z; ξ, η, ζ depend on the solution of a biquadratic equation.

2810. Let S_i be used as the symbol of the sum of i-ary products; required to prove that if i is $< n$,

$$\Sigma S_i(a_2, a_3, \dots, a_n)\frac{(a_1-a_1)(a_1-a_2)\dots(a_1-a_n)}{(a_1-a_2)(a_1-a_3)\dots(a_1-a_n)} = S_{i+1}(a_1, a_2, \dots, a_n) - S_{i+1}(a_1, a_2, \dots, a_n).$$

[For example, let $n = 3$, $i = 2$, then the theorem becomes

$$bc\frac{(a-a)(a-\beta)(a-\gamma)}{(a-b)(a-c)} + ca\frac{(b-a)(b-\beta)(b-\gamma)}{(b-a)(b-c)} + ab\frac{(c-a)(c-\beta)(c-\gamma)}{(c-a)(c-b)} = abc - a\beta\gamma, \text{ which is obviously true.}]$$

2827. If four ternary quadrics in ϕ, ψ, ω, which all vanish for one and the same set of values of the variables, be respectively equal to x, y, z, t, &c.; prove, by a simple application of the dialytic method of elimination, that these last named four quantities are connected by a cubic equation, and consequently that every cubic surface is unicursal.

2832. Prove that the curve of intersection of two right cones with parallel axes is a spherical curve, and that a third right cone may be drawn through it. Prove also that a plane circular cubic may be found such that the distances of *any* two fixed points in it from *every* point in the curve of double curvature above-mentioned shall be in a fixed linear relation.

2853. 1. Let P be a given point in a cubic. Show that, in general, three conics may be drawn having contact of the 3rd order with the cubic of P, and ordinary contact with it at some other point; but that only two conics can be drawn when P is a pluperfect point of the 1st, 2nd, or 4th order.

2. Also that, in general, twelve conics may be drawn having ordinary contact with the cubic at P, and contact of the 3rd order with it at some other point; but that only eight such conics can be drawn when P is a pluperfect point of the 1st or 2nd order.

2855. Let a, α be the initial coefficients in any two quantics ϕ and ψ. Show that the discriminant in respect to a, 1 of the resultant ϕ, ψ will contain the discriminant of ψ as a factor, and that the remaining factor will not contain α. For example, the resultant of $ax^2+2bxy+cy^2$, $\alpha x^2+2\beta xy+\gamma y^2$ may be regarded as a quadratic in a, 1, and its discriminant in respect to a is $(a\gamma-\beta^2)(c\gamma-c\beta)^2$.

2859. Let $2i-1$ distinct logarithmic waves, $y=c\log(x-a)$, be given coaxal, but with their asymptotes disposable at will along the axis. Show that, with the aid of a new wave of the same form, the $2i$ asymptotes may be so interspaced (and that, too, in a variety of ways) as that the curve which represents the combined wave shall be rectifiable as a linear function of the abscissa and ordinates of the partial waves. Show also that *in general* the new partial wave introduced to complete the system admits of $2\dfrac{\Pi(2i-1)}{\Pi i\,\Pi(i-1)}$ distinct forms, and that to each of those forms correspond $\Pi i\,\Pi(i-1)$ modes of interspacing the asymptotes.

2866. Given the simultaneous equations

$$v_{x+1}=v_x+(i^2-x)\,v_{x-1},\quad u_{x+1}=u_x+(x^2-x)\,u_{x-1}+(2x+1)\,v_{x+1};$$

prove that the general solution is of the form

$$v_x=\lambda\alpha_x+\mu\beta_x,$$

$$u_x=\tfrac{1}{2}\lambda\left\{x^2+2x+\tfrac{1}{2}(-)^x\right\}\alpha_x+\tfrac{1}{2}\mu\left\{x^2+2x-\tfrac{1}{2}(-)^x\right\}\beta_x+\nu\alpha_x+\pi\beta_x$$

(where λ, μ, ν, π are arbitrary constants), and determine the values of α_x, β_x.

2883. 1. If there be two coincident solutions of the system

$$x+y+z=a+b+c,\quad ax+by+cz=(a+b+c)^2,\quad xyz=ayz+bzx+cxy,$$

show that a curve can be formed of which a, b, c may be regarded as the trilinear coordinates, and determine its order.

2. At how many (if any) points of the curve will all three solutions coincide?

3. Is it possible for the coordinates at any point of the curve to be commensurable quantities?

2886. Write down the table (A), supposed to be indefinitely extended, in which each series of figurate numbers is taken twice over in regular succession; and from it form a symmetrical table (B) by zigzag multiplication between each two consecutive lines of figures:

(A)								(B)							
1,	1,	1,	1,	1,	1,	1,	1,	1,	1,	1,	1,	1,	1,	1,	1,
1,	2,	3,	4,	5,	6,	7,	8,	1,	1,	2,	2,	3,	4,	4,	5,
1,	2,	3,	4,	5,	6,	7,	8,	1,	2,	4,	6,	9,	12,	20,	25,
1,	3,	6,	10,	15,	21,	28,	36,	1,	2,	6,	9,	18,	24,	40,	50,
1,	3,	6,	10,	15,	21,	28,	36,	1,	3,	9,	18,	36,	60,	100,	150,
1,	4,	10,	20,	35,	56,	84,	120,	1,	3,	12,	24,	60,	100,	200,	300,
1,	4,	10,	20,	35,	56,	84,	120,	1,	4,	16,	40,	100,	200,	400,	700,
1,	5,	15,	35,	70,	126,	210,	330,	1,	4,	20,	50,	150,	300,	700,	1225,

where, *e.g.*, the line 1, 2, 6, 9, 18, 24, 40, 50.....
represents the binary products

1 . 1, 1 . 2, 2 . 3, 3 . 3, 3 . 6, 6 . 4, 4 . 10, 10 . 5.....,

and the line 1, 3, 9, 18, 36, 60, 100, 150.....
represents 1 . 1, 1 . 3, 3 . 3, 3 . 6, 6 . 6, 6 . 10, 10 . 10, 10 . 15.....

Denote the number in the rth line and sth column of table (B) by $[r, s]$, which, it will be noticed, is the same as $[s, r]$.

Let $[1, n] + [2, n-1] + [3, n-2] + \ldots\ldots + [n, 1] = P_n$,

and $\Pi n\,[1, n] + \Pi 2 \,.\, \Pi\,(n-1)\,[2, n-1] + \Pi 3 \,.\, \Pi\,(n-2)\,[3, n-2] + \ldots\ldots + \Pi n\,[n, 1] = Q_n$.

Prove that $P_{2i+1} = \dfrac{\Pi\,(2i+1)}{\Pi i\,\Pi\,(i+1)}$, $P_{2i} = \dfrac{2\Pi\,(2i-1)}{\Pi\,(i-1)\,\Pi i}$,

and that Q_n is the product of n consecutive terms of the fluctuating series

1, 4, 4, 6, 6, 8, 8, 10.....

N.B.—The term $[r, s]$ arises naturally in the theory of reducible cyclodes, or still more immediately in that of rectifiable compound logarithmic curves.

If $\left.\begin{aligned} U &= (x^2-a_1^2)\,(x^2-a_2^2)\,\ldots\ldots\,(x^2-a_r^2) \\ V &= (x^2-b_1^2)\,(x^2-b_2^2)\,\ldots\ldots\,(x^2-b_s^2) \end{aligned}\right\}$, and $Fx = U^f \,.\, V^g$,

where f, g are any two unequal quantities, and the a and b systems are to be assigned in such a manner as to be all distinct from each other, and to make $1 + (F'x \,/\, Fx)^2$ a perfect square, the number of solutions of this problem is the $[r, s]$ of the above table. Furthermore, if $f(x)$ (its leading coefficient being given) is a function of n dimensions in x^2, the total number of distinct forms of $f(x)$ which will make $1 + (f'x \,/\, fx)^2$ a perfect square is equal to what has been termed P_n in the text above. Moreover, if $n = r + s$, and the multiplicity of the roots of $f(x)$ is limited to be r (or s), the number of solutions of the Diophantine Problem is $[r, s]$.

2890. Let f, g be any two given incommensurable quantities, and U, V two rational integral functions, each of degree n in x^2, so taken as to cause the arc of the curve $y = f \log U + g \log V$ to become a linear function of x and of the logarithms of the factors of U and V. Prove that the number of curves fulfilling this condition (change by translation parallel to the axis of y being of course disregarded) is

1 if $n = 1$, $\left\{\dfrac{\Pi\,(2i-1)}{\Pi i\,\Pi\,(i-1)}\right\}^2$ if $n = 2i$, $4\left\{\dfrac{\Pi\,(2i-1)}{\Pi i\,\Pi\,(i-1)}\right\}^2$ if $n = 2i+1$;

thus giving rise to the progression 1 ; 1, 4, 9, 36, 100, 400.....

Prove also that if, instead of the degrees of U and V in x^2 being *each* n, the degree of their product in x^2 is n, everything else remaining unaltered, the number of solutions is expressed by the square roots of the preceding progression, *i.e.*, is 1; 1, 2, 3, 6, 10, 20, 35.....

2903. If i be any positive integer, and $(\Delta-3)^i u_x = \frac{\Pi(2x)}{(\Pi x)^2}$,

show that $u_x = (1, x)^{i-1} 4^x + \frac{\Pi i \,.\, \Pi(2x+2i)}{\Pi 2i \,.\, \Pi x \,.\, \Pi(x+i)}$.

N.B.—It will readily be seen that this theorem virtually contains a solution of the partial-difference-equation (not noticed by Mr. Boole),

$$u_{x,y} = u_{x-1,y} + u_{x,y-1}.$$

2906. 1. Prove that the continued fraction $1 + \cfrac{1}{p + \cfrac{1}{q + \cfrac{1}{r \ldots}}}$, where p, q, r ..... are the terms of the natural harmonic progression $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$....., is equal to $\frac{1}{4}\pi$.

2. In employing the successive convergents to the above continued fraction, explain the advantage of leaving off with a numerator 1 instead of with its denominator.

2915. If, r, ρ, R being three given unequal quantities, the expression

$$(\lambda^2 + a\lambda + b)(\lambda^2 + \alpha\lambda + \beta)(\lambda^2 + A\lambda + B)$$
$$-(\lambda^2 - a\lambda + b)(\lambda^2 - \alpha\lambda + \beta)(\lambda^2 - A\lambda + B)$$

is identical for all values of λ with the expression

$$r(2\lambda + a)(\lambda^2 + \alpha\lambda + \beta)(\lambda^2 + A\lambda + B)$$
$$+\rho(2\lambda + \alpha)(\lambda^2 + A\lambda + B)(\lambda^2 + a\lambda + b)$$
$$+R(2\lambda + A)(\lambda^2 + a\lambda + b)(\lambda^2 + \alpha\lambda + \beta),$$

prove that the six-membered equation-system which connects a, α, A, b, β, B with r, ρ, R is of the order 19.

2921. The annexed table

1	1	1	1	1	1	.....
1	3	6	10	15	21	.....
1	6	20	50	105	196	.....
1	10	50	175	496	1176	.....
1	15	105	490	1764	.....	

is symmetrical, and is obtained as follows:—

The 2nd row by adding up 3 times successively the progression 1, 0, 0, 0, 0, 0,
3rd row ,, 5 times ,, ,, 1, 1, 0, 0, 0, 0,
4th row ,, 7 times ,, ,, 1, 3, 1, 0, 0, 0,
5th row ,, 9 times ,, ,, 1, 6, 6, 1, 0, 0.

In like manner, the 6th row would be got by adding up 11 times successively the progression, 1, 10, 20, 10, 1, 0, 0; the 7th row, by adding up 13 times successively the progression 1, 15, 50, 50, 15, 1, 0, 0, and so indefinitely.

Call the number in the rth line and sth column $[r, s]$: then it is required to prove that $[r, s] = \frac{\Pi(r+s-1) \cdot \Pi(r+s-2)}{\Pi(r) \cdot \Pi(s) \cdot \Pi(r-1) \cdot \Pi(s-1)}$.

The above (r, s) table does for Reducible Cyclodes in general what the (r, s) table in Question 2886 does for the Symmetrical Class in particular.

2926. Let f, g, h, k be any four given unequal quantities, and m, n, p, q four others (to be found) such that the arc of the curve

$$y = f \log (x^2 - m^2) + g \log (x^2 - n^2) + h \log (x^2 - p^2) + k \log (x^2 - q^2)$$

is equal to $x + f \log \frac{x+m}{x-m} + g \log \frac{x+n}{x-n} + h \log \frac{x+p}{x-p} + k \log \frac{x+q}{x-q}$.

Prove that, in general, there are nine systems of values of m, n, p, q which will satisfy the above condition, but when $f+g = h+k$, or $f+h = g+k$, or $f+k = g+h$, only 7.

2934. If $s_1, s_2, s_3, s_4, s_5, s_6$ represent the sum of x, y, z, t, u, v, and of their binary, ternary, quaternary, quinary, and sextic combinations respectively, and if E stands for the symbol of emanation,

$$a \frac{d}{dx} + b \frac{d}{dy} + c \frac{d}{dz} + d \frac{d}{dt} + e \frac{d}{du} + f \frac{d}{dv},$$

prove that the resultant of s_1, Es_2, s_3, Es_4, s_5, Es_6 is the product of

$$(a-b)^9 (a+b-c-d)^2 (a+b+c-d-e-f)^{12},$$

and of the similarly formed products of powers of the analogous linear functions of a, b, c, d, e, f.

2935. If $f(x)$ be a rational integral function of x of a higher order than the second, prove that it is impossible for $(fx)^2 + (f'x)^2$ to be a perfect square unless $f(x)$ contains at least two groups of equal factors.

2941. Let N_1, N_2 be any two, the same or different, partitionments of n into 3 parts; then it may easily be verified that every combination N_1, N_2 must belong to the one or the other of the 13 subjoined types

$$\begin{array}{cccccccccccc} abc & abc & aac & aac & aac & aac & aac & aaa & aaa & aaa \\ a\beta\gamma & a\beta\gamma & a\beta\gamma & a\beta c & a\beta\gamma & a\alpha\gamma & a\gamma\gamma & a\beta\gamma & a\beta\gamma & a\alpha\gamma \end{array}$$

$$abc \quad aac \quad aaa$$

$$abc \quad aac \quad aaa.$$

Call the value of any combinations appertaining to these types 24, 20, 12, 10, 8, 6, 4, 4, 2, 2, 7, 1, 0 respectively. Required to prove that the sum of such values for all the combinations possible is $\frac{(n-1)(n-2)^2(n-3)}{12}$.

For example: if $n = 6$, the ternarian partitions of 6 are 1, 1, 4; 1, 2, 3; 2, 2, 2; and their binary combinations

$$\begin{array}{ccccccc} 1.1.4 & 1.1.4 & 1.1.4 & 1.2.3 & 1.2.3 & 2.2.2 \\ 1.1.4 & 1.2.3 & 2.2.2 & 1.2.3 & 2.2.2 & 2.2.2 \end{array}$$

belong respectively to the types

$$\begin{array}{ccccccc} a.a.c & a.a.c & a.a.a & a.b.c & a.a.a & a.a.a \\ a.a.c & a.\beta.\gamma & a.\beta.\gamma & a.b.c & a.\beta.\gamma & a.a.a \end{array}$$

the sum of whose values $= 1+8+2+7+2+0$ is $\frac{5 \cdot 4^2 \cdot 3}{12}$.

[Professor SYLVESTER remarks that this question is an example of an entirely new class of questions in partitions, brought to light by the theory of involutes to a circle, and that he is in possession of the theory for partitions of any order, instead of merely ternarian ones, as here given.]

2958. *Def.* (1.) A parallel equality means the equality of the sum of any number of quantities in a group to that of the *same number* of other quantities therein.

Def. (2.) The *genus* of a group of quantities is determined by the nature and relations of the parallel equalities it contains, or does not contain.

Ex. gr.—A group $[a, b, c, d]$ may be free from parallel equalities, or it may contain them of the form $[a = b]$ or $[a = b, c = d]$, or $[a = b = c]$, or $[a = b = c = d]$, or $[a+b = c+d]$, or $[a = b,\ a+b = c+d]$; thus a quaternary group comprises 7 genera.

Determine, according to the same rule, the number of genera in a *quinary* and *sextic* group.

2959. If $U_1, U_2, \ldots\ldots U_i$ be respectively the sums of quantics in $x_1, x_2, \ldots\ldots x_j$, the lowest quantic forming a term of U_θ being of the degree q_θ, give a short analytical proof that in the equation-system $U_1 = 0$, $U_2 = 0, \ldots\ldots U_i = 0$, the trivial solution $x_1 = 0, x_2 = 0, \ldots\ldots x_i = 0$ will represent $q_1 \cdot q_2 \ldots q_i$ of the total number of solutions; or, in other words, that $q_1 \cdot q_2 \ldots q_i$ will be the multiplicity or intensity of the trivial solution.

2997. Prove that the continued fractions derived from the form $\frac{1}{1+}\ \frac{1}{2^{-1}+}\ \frac{1}{3^{-1}+}\ \frac{1}{4^{-1}+}\ \ldots\ldots$ by cutting off from it any number of consecutive initial terms is always less than unity, but approaches indefinitely near to unity as the number of terms cut off is increased.

E.g.: $\frac{1}{4^{-1}+}\ \frac{1}{5^{-1}+}\ \frac{1}{6^{-1}+}$ *ad. inf.* is one of the fractions in question obtained by cutting off 3 of the terms from the given form, and will be less than unity in value.

3019. 1. Prove that the 27 Sextactic Points in a cubic curve may be resolved into 3 sets of 9 each; the property of each set of 9 being that the line joining any two in the same set will pass through a point of inflexion.

2. Prove that $6n+3$ points may always be so arranged that each of them shall be the point of intersection of $3n$ right lines obtained by joining two and two all but one couple of the remaining $6n+2$ points.

Ex. gr.: 15 points may be arranged so that each of them shall be the common intersection of 6 lines drawn through as many pairs of points (*i.e.* 12 points) chosen out of the 14 remaining ones; thus giving rise to $\frac{1}{2}(15 \cdot 6)$, *i.e.* 30 right lines in all.

3065. A body is attracted to a fixed centre by a force varying as any given function of the distance, and perpendicularly towards a fixed plane by a force varying as its distance from the plane; determine the motion.

3085. 1. Prove that the probability of five points taken arbitrarily within any given ellipsoid forming the angles of a reentrant polyhedron is

$$\frac{3}{356\pi}\int_0^\pi\int_0^\pi d\theta\, d\phi\, (\sin\theta)\,(\cos\theta)^4\,\mu\nu,$$

where $\mu = \{1-(\sin\theta)^2(\sin\phi)^2\}^2 + 4(\sin\theta)^2(\cos\phi)^2 - 4(\sin\theta)^4(\sin\phi)^2(\cos\phi)^2$,

and $\nu = 3 - 3(\sin\theta)^2(\sin\phi)^2 + 8\sin\theta\sin\phi$.

2. Show that this probability is the ratio of two commensurable numbers, and ascertain its value.

3106. 1. A body is bounded above and below respectively by segments of two confocal ellipsoids having a common fixed centre. Prove that it may be made to rotate about that centre so as *to remain in contact* with *any two fixed parallel* planes bisecting the two segments respectively.

2. If the lower portion of the body containing a fixed point be given, find the form of the upper portion in order that motion, as in (1), may be possible about that point and between any two fixed parallel tangent planes.

3145. *Definition.*—By the sum of any number of arrangements such as

$$a, b, \ldots l, \quad a', b', \ldots l', \quad a'', b'', \ldots l'', \ldots\ldots$$

is to be understood the arrangement $\Sigma a, \Sigma b, \ldots \Sigma l$.

If the arrangements with repetitions of the elements zero and unity i and i together (with exclusion of the arrangement in which no unity appears), be added together in every possible way; prove that the number of sums so obtained, of which each of the i elements is an even number or zero, is $2^{2^i-i-1}-1$.

E.g.: For $i = 3$, we have

$$\left.\begin{array}{l} 0,\ 0,\ 1 \\ 0,\ 1,\ 0 \\ 1,\ 0,\ 0 \\ 1,\ 1,\ 0 \\ 1,\ 0,\ 1 \\ 0,\ 1,\ 1 \\ 1,\ 1,\ 1 \end{array}\right\}$$ of which 15 sums will satisfy the required condition.

If $i = 0$, we have

$$\left.\begin{array}{l} 0,\ 1 \\ 1,\ 0 \\ 1,\ 1 \end{array}\right\}$$ of which only one (viz., the complete) sum will satisfy the condition.

It will be seen that $2^{-3-1}-1 = 15$ and $2^{4-2-1}-1 = 1$.

3480. Let there be an indefinite number of variables $x, y, z \ldots u, v, w \ldots$.

Let D_i be the number of solutions in positive integers of the equation

$$(x+x') + 2(y+y') + \ldots + (i-1)(u+u') + iv + (i+1)w + \ldots = n - im - \tfrac{1}{2}(i^2-i).$$

Prove that $D_1 - D_2 + D_3 - D_4 + D_5 \ldots$ is the number of solutions in positive integers of the simultaneous system of equations

$$x+y+z+t+\ldots = m, \quad x+2y+3z+4t+\ldots = n.$$

3427. If ϕ, ψ are quantics in x, y, each of degree μ; F a quantic in ϕ, ψ of degree m and consequently in x, y of degree $m\mu$; and if J denote the Jacobian of ϕ, ψ, that is

$$\frac{d\phi}{dx}\frac{d\psi}{dy} - \frac{d\phi}{dy}\frac{d\psi}{dx};$$

$D_{\phi,\psi}F$ the discriminant of F treated as a quantic in ϕ, ψ; $D_{x,y}F$ the discriminant of F, treated as a quantic in x, y; and if R be used as the symbol of "resultant in regard of x, y"; prove that

$$D_{x,y}F = (R_{(\phi,\psi)})^{m^2-2m} R(F, J)(D_{\phi,\psi}F)^{\mu}.$$

As a particular case of the foregoing theorem, show that the discriminant of F, any *symmetrical* quantic of an even degree in x, y, is of the form $F(1, 1)\, F(1, -1)\, \Omega^2$, where Ω is a rational integral function of the coefficients in F.

Find also what the general formula becomes, when ϕ, ψ are taken linear functions of x, y.

3506. 1. By aid of the theorem in Quest. 3480, or otherwise, prove the following theorem:—Let $x, y, z, t, \ldots$ be any system of positive integer values (zeros included) which satisfy the equation (with an unlimited number of variables) $x + 2y + 3z + 4t + \ldots = n$; and call

$$1 - x + xy - xyz + xyzt \ldots = s; \text{ then } \Sigma s = 0.$$

2. Prove also that the number of such solutions for which s exceeds zero is the coefficient of θ^n in the development of

$$\Sigma_{n=0}^{n=\infty} (-)^n \theta^{\frac{1}{2}(n^2+n)} + \Sigma_{n=-\infty}^{n=+\infty} (-)^n \theta^{\frac{1}{2}(3n^2+n)};$$

and that the number of solutions for which s equals zero is the coefficient of θ^n in the development of

$$(\theta - \theta^2)(1-\theta) + (\theta^6 - \theta^{10})(1-\theta)(1-\theta^2) + \ldots + \Sigma_{n=-\infty}^{n=+\infty} (-)^n \theta^{\frac{1}{2}(3n^2+n)}.$$

[As an example of (1), let $x + 2y + 3z + 4t = 4$; then the complete system of solutions is as follows:—

$x = 0,\ y = 2,\ z = 0,\ t = 0$; $\quad x = 2,\ y = 1,\ z = 0,\ t = 0$;
$x = 0,\ y = 0,\ z = 0,\ t = 1$; $\quad x = 4,\ y = 0,\ z = 0,\ t = 0$;
$x = 1,\ y = 0,\ z = 1,\ t = 0$;

and the corresponding values of s are 1, 1, 0, 1, −3, whose sum is zero.]

3535. 1. Euler has shown that the number of modes of composing n with i distinct numbers is equal to the denumerant (that is, the number of solutions in positive integers, zeros included) of the equation

$$x + 2y + 3z + \ldots + iw = n - \tfrac{1}{2}(i^2 + i).$$

Show more generally that the number of modes of composing n with i numbers, all distinct except the largest, which is to be always taken j times, is the denumerant of the equation

$$jx + (j+1)y + \ldots + iw = n - \tfrac{1}{2}(i - j + 1)(i + j).$$

2. Show also that, if all the partitions of n into i parts are distinct except the *least*, which is to be taken j times, then the number of such partitions is the denumerant of the equation

$$x + 2y + \ldots + (i-j)\phi + \omega = n - \tfrac{1}{2}(i-j)^2 + 3i - 1.$$

3563. *Definition.*—If, in any arrangement of integers, each of the numbers 1, 2, 3, ... up to any odd number (unity inclusive), say $2i-1$, occurs once or any odd number of times, but the even number following, say $2i$, does not occur an odd number of times, the arrangement is said to be *flushed*; if such kind of sequence does not occur, it is said to be *unflushed*.

1. Required to prove, that if any number be partitioned in every possible way, the number of unflushed partitions containing an odd number of parts is equal to the number of unflushed partitions containing an even number of parts.

Ex. gr. : The total partitions of 7 are

7; 6, 1; 5, 2; 5, 1, 1; 4, 3; 4, 2, 1; 4, 1, 1, 1; 3, 3, 1; 3, 2, 2;
3, 2, 1, 1; 3, 1, 1, 1, 1; 2, 2, 2, 1; 2, 2, 1, 1, 1; 2, 1, 1, 1, 1, 1;
1, 1, 1, 1, 1, 1, 1.

Of these, 6, 1; 4, 1, 1, 1; 3, 3, 1; 2, 2, 1, 1, 1; 1, 1, 1, 1, 1, 1, 1 alone are flushed. Of the remaining unflushed partitions, five contain an odd number of parts, and five an even number.

Again, the total partitions of 6 are

6; 5, 1; 4, 2; 4, 1, 1; 3, 3; 3, 2, 1; 2, 2, 2; 2, 2, 1, 1;
2, 1, 1, 1, 1; 1, 1, 1, 1, 1, 1;

of which 5, 1; 3, 2, 1 alone are flushed. Of the remainder, four contain an odd and four an even number of parts.

N.B.—This transcendental theorem compares singularly with the well-known algebraical one, that the total number of the *permuted* partitions of a number with an odd number of parts is equal to the same of the same with an even number.

2. Required to prove that the same proposition holds when any *odd* number is partitioned *without repetitions* in every possible way.

3621. A prime less than the square root of a number is said to be a minor prime in respect to it. A prime-partition of a number means its partition into two primes (unity reckoning as a prime), and is said to be minor if one of the parts is minor in respect to the number.

Let n be any even number, Q the total number of its prime-partitions, q the number of its minor prime-partitions, $\omega, \omega_1, \omega_2 \ldots \omega_i$ the several odd minor primes to n (exclusive of unity and all other factors of n), ν the total number of primes below n. Let ϵ, η mean 1 or 0 according as $\frac{1}{2}n$, $n-1$ respectively is or is not a prime. Required to show that

$$\frac{1}{2}\left\{\frac{(\omega_1-2)(\omega_2-2)\ldots(\omega_i-2)}{(\omega_1-1)(\omega_2-1)\ldots(\omega_i-1)}\nu + q + \epsilon - \eta\right\},$$

if integer, or if not the integer next above this quantity, is the most probable value of Q.

3651. If through $3n+1$ given points on a cubic curve a curve of the order $N+n$ be drawn, and through the remaining $3N-1$ intersections of the two curves a third one be drawn of the order N; prove that this will intersect the cubic at a fixed point.

[This point may be called the opposite of the $3n+1$ given points; it becomes Dr. SALMON's opposite, as defined by him in the *Philosophical Transactions* for 1858, when $n=1$, $N=1$, and is independent of the value of N.]

3676. If $a+2b+3c+\ldots+rl=n$; prove that, on making $\phi x = x^r - x^{r-1} - x^{r-2} \ldots -1 = 0$,

$$\Sigma \frac{\Pi n}{\Pi a\, \Pi b\, \Pi c \ldots \Pi l} = \Sigma \frac{x^{n+r-1}}{\phi' x}.$$

3708. A trifurcal ramification is a system of things or ideas, say points, consisting of terminals and joints, so conjoined, say by lines, that on taking any of the former as a root, the remaining terminals and joints are related to it and to each other after the manner of a florescence, in which, at each joint, the line, say branch, which leads up to it divides into two.

Prove that the possible schemes of trifurcal ramification with 4, 6, and 8 joints, may be obtained by cutting 3, 4, and 5 edges of a pyramid, a wedge and a cube, or truncated wedge, respectively; no regard in any ramification scheme being had to the directions or lengths of the branches.

Prove also that all the possible schemes of quadrifurcal arborescence (that, by the way, of the hydro-carbon series) may be obtained by 5 sections of the edges of an octahedron.

Prove furthermore, for each of the cases above referred to, that all the trifurcal and quadrifurcal ramifications with an inferior or intermediate number of joints, may be obtained by the process of cutting the edges as before, combined with that of freeing them from their connexion with each other at the angles of the generating figure.

3740. 1. If ϕ is any rational function of ω, investigate a simple method for determining the condition to which λ must be subject in order that $\lambda\phi + 1 = 0$ shall have one or more real roots of ω lying between 0 and 1.

2. If ϕ, ψ are any two rational functions of ω, investigate a simple method for determining the conditions to which λ, μ must be subject, in order that $\lambda\phi + \mu\psi + 1 = 0$ shall have one or more real roots of ω lying between 0 and 1.

3851. In the application of STURM's theorem to determine the total number of real roots in any equation of the nth degree, there will be a series of $(n+1)$ criteria, of which the first two are always positive, but the subsequent $(n-1)$ may any of them be positive or negative. Thus it is conceivable at first sight that 2^{n-1} cases may arise in making such application. It is required to show that the actual number of cases is less than this, being, as far as we are able to see, equal to, and certainly not in excess of, $\phi(n)$, where, according as n is odd or even,

$$\phi(n) = 2^{n-2} + \frac{\Pi(n-1)}{2\left\{\Pi\tfrac{1}{2}(n-1)\right\}^2} \quad \text{or} \quad 2^{n-2} + \frac{\Pi(n-1)}{\Pi(\tfrac{1}{2}n)\,\Pi(\tfrac{1}{2}n-1)}.$$

3907. If, in the equation $x^5 + 5ex^4 + 10e^2x^3 + 10e^2x^2 + 5\epsilon x + 1 = 0$, $e\epsilon = m$, and m is greater than 1 or less than $\frac{1}{4}$, or (abstraction made of the case where $e = \epsilon$) m is equal to 1 or equal to $\frac{1}{4}$, only one of the roots of the equation will be real. But if m is intermediate between 1 and $\frac{1}{4}$, such values may be assigned to e and ϵ as will cause three of the roots to be real, in which case the necessary (but not sufficient) condition must be satisfied, that $e\epsilon$ shall be of the form $m^{\frac{1}{2}\pm\rho}$, where $\rho < \frac{1}{10}$.

4002. If a body oscillate in the arc of a parabola under the influence of a force varying according to the law of nature tending to the focus, whose absolute value is μ, and a constant force M parallel to the axis, prove that, when μ, M are given, the time of describing any arc is a function only of the sum of the extreme focal distances and of the chord of the arc described; find the function, and show from it that the period of oscillation is a constant multiple of $\mu^{\frac{1}{2}}M^{-\frac{3}{2}}$.

[The Proposer states that the above is got with extraordinary facility by a method which he has found to lead at once to EULER's theorem on parabolic motion produced by a single force. EULER's theorem is LAMBERT's specialised. The particular case was discovered by EULER before LAMBERT, but may be proved quite independently in a single line.]

4169. If, in an equation of the fifth degree, all the invariants, but not all the coefficients, are real, prove that real roots enter into such equation

in pairs, and determine the invariantive conditions in order that an omni-reali-invariantive quintic may have 0, 2, or 4 real roots.

4506. Prove that the number of real roots in the equation

$$\lambda_1 (x+c_1) \quad + \lambda_2 (x+c_2)^m + \ldots + \lambda_n (x+c_n)^m = 0,$$

where $c_1, c_2, \ldots c_n$ form an ascending or descending series of real magnitudes, cannot exceed the number of variations of sign in the series

$$\lambda_1, \lambda_2, \ldots \lambda_n, (-1) \quad \lambda_1.$$

4529. Prove that any given equation of the 5th degree, with real coefficients, may be reduced to a trinomial form by substitutions, none of the coefficients entering into which are imaginary.

4721. Prove that every point in the plane carried round by the connecting-rod in WATTS' or any other kind whatever of 3-bar motion has in general three nodes, and that its inverse in respect to each of them is an unicircular quartic.

4771. About a circle a quadrilateral is described whose sides make the angles $\alpha, \beta, \gamma, \delta$ with any arbitrarily assumed line in their plane. Show that in general the circle is not an ellipse of minimum eccentricity inscribable in the quadrilateral, unless when

$$\Sigma (\sin \alpha)^2 (\Sigma \cos \alpha)^2 - \Sigma (\cos \alpha)^2 (\Sigma \sin \alpha)^2 = 0,$$

where it should be observed that the value of the left-hand side of the equation is a function only of the differences of the angles $\alpha, \beta, \gamma, \delta$.

5038. If a plane moves with the same two of its points always on the circumference of two circles in a fixed plane, prove that no other point in the moving plane can describe a circle except in the case when the distance between the centres of the two circles, their radii, and the distance between the two points consist of two pairs of equal quantities.

5109. If $x = a \cos(\theta+\alpha) + b \cos(2\theta+\beta) + \ldots + l \cos(n\theta+\lambda)$,

$$y = a \sin(\theta+\alpha) + b \sin(2\theta+\beta) + \ldots + l \sin(n\theta+\lambda),$$

prove (1) that the curve whose coordinates are x, y is the inverse of a unicursal curve of the order n. Also, (2) more generally, if

$$\tan\phi = e \tan\psi = e' \tan\omega = e'' \tan\theta \ldots\ldots,$$

and if, $m, n, p, q, \ldots$ being integers,

$$x = \Sigma a \cos(\alpha + m\phi + n\psi + p\omega + \ldots), \quad y = \Sigma b \sin(\alpha + m\phi + n\psi + p\omega + \ldots).$$

5152. If there be n homogeneous linear equations between $2n$ variables, and also n homogeneous linear relations between their reciprocals, prove that the resultant of the system of $2n$ equations is of the order 2^{2n-1}.

5178. x, y, z are three given rational integral biquadratic functions of t; and two systems of ratios $a : b : c$, $A : B : C$ are to be found, such that $\frac{Ax^2 + By^2 + Cz^2}{ax^2 + by^2 + cz^2}$ shall be an integral function of t, show that the problem admits in general of three, and never of more than three, solutions.

5305. *Definition.*—Six right lines along which six forces can be made to equilibrate are said to be in Statical Involution.

Prove that any six right lines lying on a ruled cubic surface are in statical involution; and, *vice versâ*, if six right lines are in statical involution, a ruled cubic surface can be made to pass through them.

5420. From the expansion of $\{\log(1+x)\}^n$, in a series according to powers of x, prove that $S_{i,j}$ [the coefficient of t^j is the developed product of $(1+t)(1+2t)\ldots\ldots(1+it)$] is divisible by every prime number greater than $j+1$ in the series $\quad i+1, i, i-1, \ldots\ldots, i-j+1$.

5593. If fx be a function of x defined by the conditions that from $x=0$ to $x=1$, $fx=0$; and from $x=1$ to $x=\infty$, $fx+f\frac{1}{2}x+f\frac{1}{3}x+\ldots$ *ad. inf.* $= x\log x$; show that (1) the curve $y=fx$ will nowhere undergo a solution of linear continuity, but will be abruptly deflected towards the axis of x whenever x is an integer; and (2) for all values of x greater than 2, fx will lie between $\log 2 \,.\, x$ and $2\log 2 \,.\, x$, and will continually increase as x increases.

5955. By a Cartesian Oval in space let us understand a curve the distances of whose points from 3 fixed points are linear functions of each other, or, which comes to the same thing, is the intersection of two surfaces of revolution, described by two plane Cartesians having a focus in common. Conversely, when two points can be found whose distances from any point in a space-Cartesian are linear functions of one another, let them be termed foci. Required to prove that the locus of such foci is a plane *curve* of the 3rd degree. It will be observed that this curve for Cartesians of double curvature is the exact analogue of the 3 foci in a straight line for plane Cartesians.

6273. Let ${}_1m m_1$, ${}_2m m_2$, ..., ${}_nm m_n$ denote any n segments of the same indefinite straight line which are the respective *ranges* of as many distinct points; P the probability of any condition of relative distances being satisfied by the group of points thus limited in range; ${}_{r,s}\mathrm{P}$, ${}_r\mathrm{P}_s$, ${}_s\mathrm{P}_r$, $\mathrm{P}_{r,s}$, respectively, the same probability when the r^{th} and s^{th} ranges are severally contracted to the pairs of points ${}_rm_sm$, ${}_rmm^s$, ${}_smm^r$, m_rm_s; and let

$$\mathrm{F}(r,s) = \frac{({}_rm_sm)^2\,{}_{r,s}\mathrm{P} + (m_rm_s)^2\,\mathrm{P}_{r,s} - ({}_rmm_s)^2\,{}_r\mathrm{P}_s - (m_r{}_sm)^2\,{}_s\mathrm{P}_r}{[{}_rm_smm_s]\,[{}_sm_rmm_r]\,({}_rmm_r)\,({}_smm_s)},$$

where each of the two sets of 3 points within the square brackets indicates $+1$ when the middle point *written down* lies between the other two, and (-1) in the contrary cases; then (1) prove that $\mathrm{P} = \frac{\Sigma \mathrm{F}(r,s)}{n(n-1)}$; (2) extend the theorem to the more general case where $\nu_1, \nu_2, \ldots \nu_n$ points appertain to 1st, 2nd, ... nth ranges respectively; and (3) apply the extended theorem to determine the chance of any *defined* anharmonic ratio formed by two pairs of points limited to two given collinear ranges constituting an infra-harmonic or supra-harmonic group.

6306. Given the systems of equations

$$x_1(x_1+x_2)(x_1+x_3)\ldots(x_1+x_n) = \mathrm{C}_1(x_1-x_2)(x_1-x_3)\ldots(x_1-x_n),$$
$$x_2(x_2+x_1)(x_2+x_3)\ldots(x_2+x_n) = \mathrm{C}_2(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n),$$
$$\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots$$
$$x_n(x_n+x_1)(x_n+x_2)\ldots(x_n+x_{n-1}) = \mathrm{C}_n(x_n-x_1)(x_n-x_2)\ldots(x_n-x_{n-1});$$

prove that the number of solutions cannot exceed $1^2 \,.\, 3^2 \,.\, 5^2 \ldots (n-1)^2$ or $1^2 \,.\, 3^2 \,.\, 5^2 \ldots (n-2)^2$, according as n is even or odd.

7143. If $\qquad \log \mathrm{F}(x,y) = \Sigma \log\left(x - 2\cos\frac{2\lambda\pi}{\kappa}y\right),$

where λ is to assume all values prime to κ and not exceeding $\frac{1}{2}(\kappa-1)$; prove that, when x, y are relative primes, $\mathrm{F}(x,y)$ can have no *prime* factors other than divisors of κ or of the form $\kappa i \pm 1$.

7164. The late Professor DE MORGAN has observed that the number of modes of composing n with the numbers 1, 2, 3, is the nearest integer to $\frac{1}{12}(n+3)^2$. Show universally that the number of ways of compounding n with 3 numbers p, q, r, of which each is prime to the others, is the nearest integer to $\frac{(2n+\alpha+\beta+\gamma)^2}{8\alpha\beta\gamma}$.

7189. Sum the series

$$1+(x-i)+\frac{(x-2i)(x-2i-1)}{1.2}+\frac{(x-3i)(x-3i-1)(x-3i-2)}{1.2.3}+\ldots\ldots$$

7219. Prove (1) that a succession of integers, no one of which is a multiple of any one or the sum of multiples of any two of its antecedents, cannot consist of an infinite number of terms; and show (2) how to extend the theorem to the case of a succession of integer duads, or more generally to a succession of n-ads.

7300. Let $t(x)$ denote the sum of the totients of all the integers not exceeding a general quantity x [the totient of an integer i, denoting the number of numbers less than i and prime to it.]

If $\quad X = t(x)-t(\frac{1}{2}x)+t(\frac{1}{3}x)-t(\frac{1}{4}x)+t(\frac{1}{5}x)\ldots$ (*ad. inf.*),

prove the inequalities $X > \frac{1}{4}(x-1)^2$, $X =$ or $> \frac{1}{4}(x+1)^2$.

[Professor SYLVESTER remarks that, by means of the above inequalities, a method similar to that employed by him in extending TCHEBICHEFF'S theory (in Vol. III. of *American Journal of Mathematics*), leads to the determination of very close exact algebraical limits to the value of $t(x)$, and shows that $\frac{3}{\pi^2}x^2$ is a fair average value of $t(x)$, which, it should be noticed, when x is an integer, is the same as the number of distinct fractional quantities that can be formed by means of all the integers from 1 to x inclusive. When x is an even integer, $X = \frac{1}{8}x^2$, and when x is an odd integer, $X = \frac{1}{4}(x+1)^2$.]

7424. Prove that, (1) when $4m+3$ is a composite number, the number of numbers not exceeding m prime to $4m+3$ is always even; and (2)

$$\Sigma_{t=2n}^{t=1}\left(\frac{t}{8n+3}\right)=0,\quad \Sigma_{t=4n+3}^{t=2n+2}\left(\frac{t}{8n+7}\right)=0,$$

where, in general, JACOBI'S generalization of the Legendrian symbol (m/p) is denoted by (t/m), with the more modern superadded convention that $(m/p)=0$ when m is not prime to p.

7480. *Definition.*—A square matrix is said to be a simple null when its determinant is zero, a binary null if all its first minors are zero, a ternary null if all its second minors are zero, and, supposing its order to be i, an i-ary (or absolute) null if each element is zero. In these several cases the matrix will be said to be of the nullity 1, 2, 3, ... i, and the nullity of a matrix whose determinant is finite will of course be said to be zero.

(1) Prove that the nullity of the product of any number of matrices of the same order cannot be less than the nullity of any *one*, nor greater than the sum of the nullities of *all* of them.

(2) Augment each term in the principal diagonal of a square matrix of order i by λ, and let $\lambda_1, \lambda_2, \ldots \lambda_i$ be the roots of the determinant of the matrix so augmented. Prove that the nullity of the product of the matrices $(m+\lambda_1)^{\alpha_1}(m+\lambda_2)^{\alpha_2}\ldots(m+\lambda_\theta)^{\alpha_\theta}$ is exactly θ^{α_θ}, where it is to

be understood that $m+\lambda_\theta$ means the matrix obtained by adding λ_θ to each term in the principal diagonal, if m and $a_1, a_2, \ldots a_\theta$ are any integers whatever. [When $\theta = i$, this theorem becomes identical with one given, but without proof, by Professor CAYLEY, in his Memoir on Matrices, in the *Philosophical Transactions* for 1858.]

7486. If $u_0 > 1$, $u_1 = u_0 + k \log u_0$, $u_2 = u_1 + k \log u_1$, $u_3 = u_2 + k \log u_2$, ...; prove that, when u_2 becomes sufficiently great, the principal term in the function which represents the sum of the reciprocals of $u_0, u_1, u_2, \ldots u_x$, will be $k^{-1} \log . \log . u_x$.

7596. Extending the meaning of the word Bicorn to signify the general curve defined by the equations $x = \frac{c^3}{\phi^4 + \phi^3}$, $y = \frac{c^2(2\phi+3)}{\phi^3+\phi^2}$, show (1) that the limiting form of the curve, as c converges towards zero, is a semicubical parabola and its axis extended indefinitely in both directions, forming together a sort of Trident; and (2) explain the mode in which the geometrical passage of the double-horned to the tridental form takes place, and specify in the Trident the morphological equivalents of the respective four branches of the general curve. [The result can be obtained without going to the general equation; for, when c is infinitesimal, for x, y to be finite we must have $\phi \equiv 0$ or $\phi \equiv -1$. When $\phi \equiv 0$, $x = \frac{c^3}{\phi^3}$, $y = 3\frac{c^2}{\phi^2}$, $27x^2 = y^3$; when $\phi = -1$, $x = -\frac{c^3}{\phi+1}$, $y = \frac{c^2}{\phi+1}$, $x - cy = 0$, and ultimately $x = 0$, hence $x(27x^2 - y^3) = 0$ is the limiting (Tridental) form.]

7663. Let $p+q$ points be required to satisfy any prescribed condition of form, and (1) let ϕ be the probability of the form when p of the points lie in an area bounded by any plane contour, and q of them in a right line cutting the contour; (2) let O be any point in the right line not external to the contour, and ω the probability of the form when one of the p points is limited to the contour itself, with a probability for any point thereof varying as the perpendicular from O upon the tangent at that point; (3) let u be the probability of the form when one of the q-points is fixed at the end of the right line distant l from O, and v the same when it is fixed at the other end distant m from O; then prove that

$$\phi = \left(2p\omega + \frac{lq}{l+m}u + \frac{mq}{l+m}v\right) + 2p + q.$$

7668. If m be any positive integer, and a, b, c any real quantities whatever, prove that the equation

$$a\left(1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots + \frac{x^m}{1.2.3\ldots m}\right)$$
$$+ b\left(1 - x + \frac{x^2}{1.2} - \frac{x^3}{1.2.3} + \ldots \pm \frac{x^m}{1.2.3\ldots m}\right) + c = 0$$

can never have more than two real roots.

7740. If $px^2 + qx + r = 0$ be an equation in matrices of the 2nd order, or (if anyone pleases to say so) in quaternions, it follows from HAMILTON'S investigations (see Lectures on Quaternions, pp. 631, 632) that this equation has in general 6 roots, reducible to 4 if a certain single ordinary equation is satisfied by the elements of the matrices p, q, r. Show that,

if this condition is satisfied for any given system of matrices p, q, r it will also be satisfied by any three homogeneous linear functions of p, q, r.

[Professor SYLVESTER recommends the student to prove that the condition referred to is found by equating to zero the discriminant in respect to u, v, w of the determinant to $up+vq+wr$, from which its invariability when p, q, r undergo linear substitution is an immediate consequence. It will, of course, be understood that if

$$p = \begin{matrix} a, & b \\ c, & d \end{matrix} \quad q = \begin{matrix} a', & b' \\ c', & d' \end{matrix} \text{ and } r = \begin{matrix} a'', & b'' \\ c'', & d'' \end{matrix} \quad up+vq+wr \text{ means}$$

$$au+a'v+a''w, \quad bu+b'v+b''w, \quad cu+c'v+c''w, \quad du+d'v+d''w.]$$

7867. If x, y, z be positive quantities connected together by the equation $x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}+\mu x^{\frac{1}{2}}.y^{\frac{1}{2}}.z^{\frac{1}{2}}=0$, show that the ratio of the greatest of them to the one next in order of magnitude cannot exceed a certain limit (a discontinuous function of μ), and give rules for determining the value of this limit as μ passes from positive to negative infinity.

7889. If, by the developant of ω^2, matrices of order ω be understood in general the determinant to the square matrix formed by *developing* each of them into a line of ω^2 terms, and under- or over-writing these lines in regular order to one another, and if by the quantic associated to i such matrices, say $m_1, m_2, \ldots m_i$ be understood the quantic of order ω, in $x_1, x_2, \ldots x_i$, which is the determinant to the matrix

$$x_1 m_1 + x_2 m_2 + \ldots + x_i m_i ;$$

prove that the invariant of the quantic associated to four matrices of the second order is (at all events to a numerical multiplier *près*) the square of their developant.

7959. State under what circumstances the fourth real focus of a bicircular quartic whose other three real foci lie in a right line becomes indeterminate, and find what becomes of the bicircular quartic in that case. [The norm of

$$(b-c)\left[(x-a)^2+y^2\right]^{\frac{1}{2}}+(c-a)\left[(x-b)^2+y^2\right]^{\frac{1}{2}}+(a-b)\left[(x-c)^2+y^2\right]^{\frac{1}{2}}$$

is a constant envelope of y^2. Thus, when A, B, C are the three given foci and AB, BC, CA the three values of n, l, m, in the equation $\Sigma b\,(\mathrm{PA})^2 = 0$, the bicircular quartic, instead of becoming merely a circular cubic by virtue of $l+m+n=0$, becomes the indefinite line ABC (repeated) and the line at infinity. This is easily verifiable geometrically.]

7999. Prove that, (1) if x, y, z, t be linear functions of a single variable (the coefficients of connexion being real), the equation

$$x^{2m}+y^{2m}+z^{2m}-t^{2m}=0$$

cannot have more than two real roots, and the equation

$$x^{2m}+y^{2m}-z^{2m}-t^{2m}=0$$

not more than four real roots (the case of all the roots being equal in each case excepted); (2) the chance of there being not more than two real roots in the second equation must be considerably greater than 2 to 1; and (3) the number of real roots in the equation

$$\pm u_1^m \pm u_2^m \pm \ldots \pm u_n^m = 0,$$

where $u_1, u_2 \ldots u_n$ are real linear functions of x, can never exceed twice the

inferior number of like signs when m is even, nor such number increased by unity when m is odd.

8184. Prove that (1) the equations between two corresponding points in two systems homologically related, may be put under the form
$x' = \rho x + f(ax+by+cz)$, $y' = \rho y + g(ax+by+cz)$, $z' = \rho z + h(ax+by+cz)$;
and (2) if two homographic systems of points connected by the equations $x' = Ax$, $y' = By$, $z' = Cz$, are moved into homology, the condition that the pole shall be contained in the axis of homology is expressed by

$$[AB(A-C)(B-C)p^2 + BC(B-A)(C-A)q^2 + CA(C-B)(A-B)r^2]^3 + A^2B^2C^2(A-C)(B-C)p^2 + (B-A)(C-A)q^2 + (C-B)(A-B)r^2]^3 = 0,$$

where p, q, r are the lengths of the sides of the fundamental triangle.

8242. If by a Simplicissimum of the nth order be understood a figure in space of n dimensions formed by the indefinite protraction of the series of which a linear segment, a triangle, and a pyramid are the three first terms, prove that (1), when each edge is unity, the squared content is $\frac{(n+1)}{2^n(1.2.3\ldots n)^2}$, and hence (2) deduce that the Cayleyan Persymmetrical Invertebrate Determinant of the squared edges by which such squared content is imaged must be diminished in the ratio of negative unity to $(-2)^n(1.2.3\ldots n)^2$, in order that it may represent its absolute value. *Ex. gr.*, the determinant

$$\begin{vmatrix} . & (ab)^2 & (ac)^2 & 1 \\ (ba)^2 & . & (bc)^2 & 1 \\ (ca)^2 & (cb)^2 & . & 1 \\ 1 & 1 & 1 & . \end{vmatrix}$$

images the squared content of a triangle whose edges are (ab), (ac), (bc), for the triangle vanishes when this determinant vanishes, but the actual value of the squared content is this determinant diminished in the ratio of negative unity to $2^2(1.2)^2$, *i.e.*, multiplied by $-\frac{1}{16} = (2)$.

8275. If M represent the matrix

$$\begin{matrix} \lambda+fa+f'a', & fb+f'b', & fc+f'c', & fd+f'd'; \\ ga+g'a', & \lambda+gb+g'b', & gc+g'c', & gd+g'd'; \\ ha+h'a', & hb+h'b', & \lambda+hc+h'c', & hd+h'd'; \\ ka+k'a', & kb+k'b', & kc+k'c', & \lambda+kd+k'd'; \end{matrix}$$

show that, if $fa+gb+hc+kd = f'a'+g'b'+h'c'+k'd' = s$,
and $fa'+gb'+hc'+kd' = f'a+g'b+h'c+k'd = 0$,
then the determinant of M is equal to $(\lambda+s)^2$, and all the first minors of M will contain $(\lambda^2+s\lambda)$ as a factor.

8347. If U represent the recurrent form

$$ax^n + bx^{n-1} + cx^{n-2} + \ldots + cx^2 + bx + a,$$

and V the recurrent form *gauche*, $ax^n + \beta x^{n-1} + \gamma x^{n-2} + \ldots + \gamma x^2 - \beta x - a$ [which necessitates that, when n is even, the middle coefficient in V shall be zero], and if U, V have no common algebraical factor, then the discriminant of $U + iV$, which is obviously real, will be positive or negative. Prove that (1) if it is positive, the number of roots in the equation $U + iV = 0$ of the form $\cos\alpha + i\sin\alpha$ will, to the modulus 4, be congruent

to 0 or 1 (according as n is even or odd); (2) if it is negative, the number of such roots will, to the same modulus, be congruent to 2 or 3 (according as n is even or odd); and (3) all the roots not of the form $\cos\alpha + i\sin\alpha$, will be associated in pairs, those of the same pair being respectively

$$\rho(\cos\omega + i\sin\omega), \quad \rho^{-1}(\cos\omega - i\sin\omega).$$

***** Professor SYLVESTER wishes it to be here stated that the order of of these Questions indicates the order of their publication, and not that in which they were sent to the Editor. Between the date of receipt and the date of publication an interval of several years has, sometimes, necessarily elapsed.**

APPENDIX II.,

SOLUTIONS OF SOME OLD QUESTIONS.

By Asûtosh Mukhopâdhyây, M.A., F.R.A.S.

3121. (By Professor Genese, M.A.)—ABC, A′B′C′, are two triangles in perspective; P, Q, R, points on their axes of homology. If AP, BQ, CR cointersect, so will A′P, B′Q, C′R.

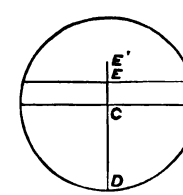
Solution.

Since the triangles ABC, A′B′C′ are in perspective, it follows that the lines joining corresponding vertices are concurrent, and the points of intersection of the corresponding sides are collinear. Now, consider another triangle PQR, such that AP, BQ, CR are concurrent, which shows that the triangles PQR and ABC are in perspective; hence, each of the triangles A′B′C′, PQR is in perspective with the triangle ABC, and therefore they are in perspective with each other (Townsend's *Geometry*, § 380), which at once shows that A′P, B′Q, C′R also cointersect; and if, as a particular limiting case, we take the points P, Q, R on the axis of homology of the triangles ABC, A′B′C′, the triangle PQR is of evanescent area, and the theorem in question thus follows with ease.

3171. (By the Editor.) — A sphere floats half-immersed within a cylinder of twice its radius partly full. If it be subjected to a small vertical displacement, show that the time of a complete oscillation is $2\pi(a/2g)^{\frac{1}{2}}$, where a is the radius of the sphere; the alteration of pressure, produced by the water being in motion, may be neglected.

Solution.

Let m be the mass of the sphere, ρ its density, and V the volume of CD; then, since the sphere is initially in equilibrium, we have mg = the weight of fluid displaced = $g\rho V$. Let the radius of the cylinder be n times that of the sphere, and suppose that there is a small vertical displacement, viz., that the sphere is depressed through a small height CE = z; then the new portion of the body, which is thus sunk into the fluid, may be considered to be



a thin cylinder. Now, it is clear that the quantity of fluid thus displaced will raise the height of the surface to some point E′, such that, if $EE' = h$, we have $\pi a^2 z = [\pi (na)^2 - \pi (a)^2] h$, which gives $z = (n^2 - 1) h$. Hence we see that, if the body is depressed through a depth z, it *actually* sinks through a depth $(h + z) = n^2 z / (n^2 - 1)$. Now, the moving force upwards on the body is the difference between the weight of the body and the weight of the fluid displaced $= g\rho A (h + z)$, where A is the area of the plane of floatation; hence the equation of motion is

$$m \cdot \frac{d^2 (CE)}{dt^2} = g\rho A \cdot \frac{n^2}{n^2 - 1} z,$$

which, since $CE = -z$, becomes $\quad m \frac{d^2 z}{dt^2} + g\rho A \cdot \frac{n^2}{n^2 - 1} z = 0.$

Hence, because $m = \rho V$, $V = \frac{4}{3}\pi a^3$, $A = \pi a^2$, the equation becomes

$$\frac{d^2 z}{dt^2} + \frac{3n^2}{2(n^2 - 1)} \cdot \frac{g}{a} \cdot z = 0,$$

and the time of a complete oscillation is, therefore,

$$2\pi \left\{ \frac{2(n^2 - 1)}{3n^2} \cdot \frac{a}{g} \right\}^{\frac{1}{2}}.$$

If $n = 2$, we have $T = 2\pi (a / 2g)^{\frac{1}{2}}$.

3370. (By Hugh McColl. Extension of Quest. 3292.)—From any point C, within or without any given boundary, a straight line CP is drawn in a random direction, meeting the boundary at the point P. Show that the average area of the circle of which the variable CP is the radius, is equal to the area enclosed by the given boundary; it being understood that the circle is positive when an infinitesimal increase of CP (its direction remaining unchanged) would place the point P without the boundary; negative when an infinitesimal increase of CP would place P within the boundary; and zero, when the line drawn at random does not meet the boundary, and the point P is imaginary; in this last case, C must, of course, be without the given boundary.

Solution.

Take C to be the origin, and employ polar coordinates, so that the point P, which is any point on the boundary, may be represented by (r, θ). Then, the average area of the circle is

$$M = \left(\int_0^{2\pi} \pi r^2 d\theta \right) \Big/ \left(\int_0^{2\pi} d\theta \right) = \left(\pi \int_0^{2\pi} r^2 d\theta \right) \Big/ 2\pi = \tfrac{1}{2} \int_0^{2\pi} r^2 d\theta$$

= Area enclosed by the given boundary.

It is self-evident that this very comprehensive theorem holds for any number of detached areas with separate boundaries. [For a solution of Quest. 3292, see Vol. xv., p. 104.]

3372. (By Professor GENESE, M.A.)—Two similar ellipses are placed so that the major axis of either coincides with the minor of the other; prove that the lines joining the common centre to the common points are perpendicular to the common tangents.

Solution.

The equations of the two conics, similar, concentric, and placed as directed in the question, are

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{\lambda^2 b^2}+\frac{y^2}{\lambda^2 a^2}=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots(1, 2).$$

Subtracting (1) from (2), we have

$$\left(\frac{y}{x}\right)^2=\frac{a^2-\lambda^2 b^2}{\lambda^2 a^2-b^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3),$$

which represents loci passing through the common points of (1) and (2), viz., they are obviously the chords of intersection, which, accordingly, pass through the origin. Again,

$$y=mx+(b^2+a^2m^2)^{\frac{1}{2}}, \quad y=mx+\lambda\,(a^2+b^2m^2)^{\frac{1}{2}} \quad \ldots\ldots\ldots\ldots(4, 5)$$

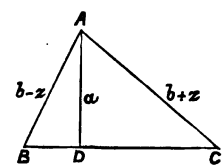
represent any two tangents to (1) and (2). In order that these may coincide, we must have

$$m^2=\frac{a^2\lambda^2-b^2}{a^2-b^2\lambda^2},$$

and, for this value of m, (4) is evidently perpendicular to (3), which is the theorem in question. [For another solution, see Vol. XLIII., p. 41.]

3377. (By Professor ARTEMAS MARTIN, M.A., Ph.D., LL.D.)—The sum of two sides of a triangle is $2b$, and the perpendicular from the angle included by these sides upon the third side is a. Required the average area of the triangle.

Solution.



Let one of the sides be $(b-z)$, so that the other is $(b+z)$, z being a variable parameter. Then it would *at first sight* appear as if the limits of z were $+b$, and $-b$; but a moment's consideration shows that, as a remains constant in magnitude, $(b-z)$ can never be less than a; and the same argument applies to the other side, so that the limits of z are given by $b+z=a$, $b-z=a$, or $z=a-b$, $z=b-a$. Now, the length of the base is evidently $\mathrm{BD}+\mathrm{DC}=\{(b-z)^2-a^2\}^{\frac{1}{2}}+\{(b+z)^2-a^2\}^{\frac{1}{2}}$. Therefore, if Δ be the area of the triangle, and M the average area sought, we have

$$\frac{2\Delta}{a}=\{(b-z)^2-a^2\}^{\frac{1}{2}}+\{(b+z)^2-a^2\}^{\frac{1}{2}},$$

and

$$\mathrm{M}=\left(\int_{b-a}^{a-b}\Delta\,dz\right)\Big/\left(\int_{b\ \ a}^{a-b}dz\right),$$

whence

$$2\,(a-b)\,\mathrm{M}=\int_{b-a}^{a-b}\Delta\,dz,$$

and $$\frac{4(a-b)}{a}\,\mathrm{M} = \int_{b-a}^{a-b} \frac{2\Delta}{a}\,dz$$

$$= \int_{b-a}^{a-b} \left\{(b-z)^2 - a^2\right\}^{\frac{1}{2}} dz + \int_{b-a}^{a-b} \left\{(b+z)^2 - a^2\right\}^{\frac{1}{2}} dz.$$

Transform the integrals on the right-hand side by the substitutions $b-z=u$, $b+z=v$, which give for u the limits $(2b-a,\ a)$, and for v the limits $(a,\ 2b-a)$. Therefore

$$\frac{4(a-b)}{a}\,\mathrm{M} = \int_{2b-a}^{a} (u^2-a^2)^{\frac{1}{2}}\,du + \int_{2b-a}^{a} (v^2-a^2)^{\frac{1}{2}}\,dv,$$

and, as the value of a definite integral is independent of what symbol we may use for the variable, we have

$$\frac{4(a-b)}{a}\,\mathrm{M} = 2\int_{2b-a}^{a} (z^2-a^2)^{\frac{1}{2}}\,dz, \quad \text{or} \quad \frac{2(b-a)}{a}\,\mathrm{M} = \int_{a}^{2b-a} (z^2-a^2)^{\frac{1}{2}}\,dz$$

$$= \left\{\tfrac{1}{2}z\,(z^2-a^2)^{\frac{1}{2}} - \tfrac{1}{2}a^2 \log\left[z + (z^2-a^2)^{\frac{1}{2}}\right]\right\}_{a}^{2b-a}$$

$$= (2b-a)\left[b\,(b-a)\right]^{\frac{1}{2}} - \tfrac{1}{2}a^2 \log\left\{2b-a+2\left[b\,(b-a)\right]^{\frac{1}{2}}\right\} + \tfrac{1}{2}a^2 \log a$$

$$= (2b-a)\left[b\,(b-a)\right]^{\frac{1}{2}} - \tfrac{1}{2}a^2 \log\left\{\frac{b^{\frac{1}{2}} + (b-a)^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right\}^2$$

$$= (2b-a)\left[b\,(b-a)\right]^{\frac{1}{2}} - a^2 \log \frac{b^{\frac{1}{2}} + (b-a)^{\frac{1}{2}}}{a^{\frac{1}{2}}},$$

whence, finally, $$\mathrm{M} = \tfrac{1}{2}a^2 \left\{\left(2\,\frac{b}{a} - 1\right)\left(\frac{b}{b-a}\right)^{\frac{1}{2}} - \frac{a}{b-a} \log \frac{b^{\frac{1}{2}} + (b-a)^{\frac{1}{2}}}{a^{\frac{1}{2}}}\right\}.$$

Since it is geometrically evident that $(b-z)$ and $(b+z)$ are, each of them, greater than a, it follows that b is greater than a, which shows that M is always real, as is to be expected. If we put $b = a\lambda$, $a^2 = \mu$, the result may be written

$$\mathrm{M} = \tfrac{1}{2}\mu \left\{(2\lambda - 1)\left(\frac{\lambda}{\lambda-1}\right)^{\frac{1}{2}} - \left(\frac{1}{\lambda-1}\right)^{\frac{1}{2}} \log\left[\lambda^{\frac{1}{2}} + (\lambda-1)^{\frac{1}{2}}\right]\right\}.$$

We now proceed to give an interesting geometrical interpretation of the above result, viz., it has been proved that

$$\frac{2(b-a)}{a}\,\mathrm{M} = \int_{a}^{2b-a} (z^2-a^2)^{\frac{1}{2}}\,dz,$$

and it is a well-known theorem, familiar to every student of analytic geometry, that if we take the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$,

the area between the curve, and the transverse axis, from the vertex to the ordinate $x = 2b-a$, is given by

$$\mathrm{A} = \int y\,dx = \frac{b}{a}\int_{a}^{2b-a} (x^2-a^2)^{\frac{1}{2}}\,dx,$$

whence we conclude that

$$\mathrm{A} = \frac{b}{a} \cdot \frac{2(b-a)}{a} \cdot \mathrm{M}, \quad \text{or} \quad \mathrm{M} = \frac{a^2}{2b\,(b-a)}\,\mathrm{A},$$

which shows that the mean value of the triangle bears a constant ratio to a portion of the area of the hyperbola, of which the semi-axes are equal to the given altitude and semi-sum of sides respectively. It is further to

be noted that this ratio is independent of a, b, viz., it is the reciprocal of $2\lambda(\lambda-1)$. This question is specially interesting, owing to the difficulty in the determination of the limits noticed above. It is clear that the limits of z cannot be $(+b, -b)$, as, in that case, *at the limits*, the triangle would vanish, and would thus have its altitude infinitesimal instead of the finite quantity a; in fact, if we work with the limits $(+b, -b)$, we find

$$\mathrm{M} = \frac{a}{2b}\int_0^{2b} (z^2 - a^2)^{\frac{1}{2}}\, dz,$$

and the integral on the right-hand side is imaginary from $z = 0$ to $z = a$, as is sufficiently obvious, analytically from actual calculation, geometrically from the theory of the hyperbola; but, as all the triangles are real, their mean value cannot possibly be imaginary, and this shows that the limits $(+b, -b)$ *must* be wrong.

[Another solution is given on p. 84 of this volume.]

3713, 5216, & 6706. (By Sir Robert Ball, F.R.S.)—If $x > y\cot a$, prove that the expression $\sec^2\frac{1}{2}a\,(x\cos\frac{1}{2}a + y\sin\frac{1}{2}a)$ will give the value of $(x^2+y^2)^{\frac{1}{2}}$ correct to within 100 $\tan^2\frac{1}{2}a$ per cent. of the whole.

Solution.

Since $a^2 + b^2 > 2ab$, whenever a and b are unequal, we have

$$x^2\sin^2\tfrac{1}{2}a + y^2\cos^2\tfrac{1}{2}a > 2\,.\,x\sin\tfrac{1}{2}a\,.\,y\cos\tfrac{1}{2}a,$$

provided that x be not equal to $y\cot\frac{1}{2}a$. Therefore

$$x^2+y^2 > x^2\cos^2\tfrac{1}{2}a + y^2\sin^2\tfrac{1}{2}a + 2xy\sin\tfrac{1}{2}a\cos\tfrac{1}{2}a > (x\cos\tfrac{1}{2}a + y\sin\tfrac{1}{2}a)^2,$$

therefore $$(x^2+y^2)^{\frac{1}{2}} > x\cos\tfrac{1}{2}a + y\sin\tfrac{1}{2}a.$$

Let $$z_1 = (x^2+y^2)^{\frac{1}{2}}, \quad z_2 = (x\cos\tfrac{1}{2}a + y\sin\tfrac{1}{2}a)\sec^2\tfrac{1}{2}a.$$

Then $$(x\cos\tfrac{1}{2}a + y\sin\tfrac{1}{2}a)\sec^2\tfrac{1}{2}a < (x^2+y^2)^{\frac{1}{2}}\sec^2\tfrac{1}{2}a,$$

or $$z_2 < z_1\sec^2\tfrac{1}{2}a < z_1 + z_1\tan^2\tfrac{1}{2}a,$$

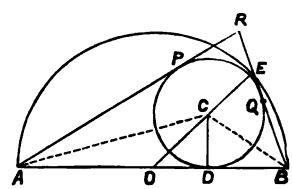
therefore $$z_2 - z_1 < z_1\tan^2\tfrac{1}{2}a,$$

which at once shows that z_2 gives the value of z_1 correct to within 100 $\tan^2\frac{1}{2}a$ per cent. of the whole, which is just the theorem in question.

4021. (By the Editor.)—From the ends of the base of a semi-circle draw tangents to any circle inscribed in it, and find the equation and area of the locus of their intersection.

Solution.

Let AEB be the given semicircle, AB its diameter, O its centre; PED is the circle inscribed in this semicircle, touching the base at D, and the arc at E, so that if C be its centre OCE is a straight line. Draw AP, BQ tangents to the inner circle, intersecting in R; we have to determine the locus of R. Join CA, CB, CD. Take O to be the



origin of coordinates, OB as the axis of x, and let $OA = OB = a$. Let (α, β) be the coordinates of C; then $OD = \alpha$, and $CE = CD = \beta$; since $OE = OB = a$, we have $OC = OE - CE = a - \beta$; hence, from the geometrical condition $OC^2 = OD^2 + DC^2$, we get $(a-\beta)^2 = \alpha^2 + \beta^2$, implying $\beta = (a^2 - \alpha^2)/2a$. We have next to obtain the equations of AP, BQ; since B is $(a, 0)$, any line through it may be written $y = m(x-a)$, where m is the tangent of the angle of inclination of the line to the axis of x; hence, this equation will represent BQ, if

$$m = \tan OBQ = \tan 2DBC = \tan 2\theta, \text{ say.}$$

Now, because $\tan\theta = CD/DB$, we have

$$\tan\theta = \frac{\beta}{a-\alpha}, \quad \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\beta(a-\alpha)}{(a-\alpha)^2 - \beta^2}$$

so that the equation of BQ is $y = \dfrac{2\beta(a-\alpha)}{(a-\alpha)^2-\beta^2}(x-a)$ .....(1).

Again, the coordinates of A being $(-a, 0)$, and

$$\tan CAD = \frac{CD}{DA} = \frac{\beta}{a+\alpha},$$

the equation of AP is similarly found to be

$$y = \frac{2\beta(a+\alpha)}{(a+\alpha)^2-\beta^2}(x+a) \text{ .....(2).}$$

We have, now, to find the locus of the point of intersection of (1) and (2), when the variable parameters α, β are connected by the relation

$$\beta = \frac{a^2-\alpha^2}{2a} \text{ .....(3).}$$

Equating the values of y from (1) and (2), we get

$$\frac{2\beta(a-\alpha)}{(a-\alpha)^2-\beta^2}(x-a) = \frac{2\beta(a+\alpha)}{(a+\alpha)^2-\beta^2}(x+a),$$

or $$x\left\{\frac{a-\alpha}{(a-\alpha)^2-\beta^2} - \frac{a+\alpha}{(a+\alpha)^2-\beta^2}\right\} = a\left\{\frac{a-\alpha}{(a-\alpha)^2-\beta^2} + \frac{a+\alpha}{(a+\alpha)^2-\beta^2}\right\},$$

which finally reduces to $x = \dfrac{a^2}{\alpha} \cdot \dfrac{a^2-\alpha^2-\beta^2}{a^2-\alpha^2+\beta^2}$ .....(4).

This easily gives $x - a = \dfrac{a(a+\alpha)\{(a-\alpha)^2-\beta^2\}}{\alpha(a^2-\alpha^2+\beta^2)}$ .....(5),

which, being substituted in (1), gives $y = \dfrac{2a\beta(a^2-\alpha^2)}{\alpha(a^2-\alpha^2+\beta^2)}$ .....(6).

Since, from (3), we have $a^2 - \alpha^2 = 2a\beta$, (4) and (6) reduce to

$$x = \frac{a^2}{\alpha} \cdot \frac{2a-\beta}{2a+\beta}, \quad y = \frac{4a^2\beta}{\alpha(2a+\beta)},$$

which, on substituting for β from (3), are found to lead to

$$x = \frac{a^2}{\alpha} \cdot \frac{3a^2+\alpha^2}{5a^2-\alpha^2}, \quad y = \frac{4a^2}{\alpha} \cdot \frac{a^2-\alpha^2}{5a^2-\alpha^2} \text{ .....(7, 8).}$$

We thus see how the coordinates of any point on the locus of R can be expressed in terms of a single parameter α. This system furnishes us with every information regarding this locus.

To obtain the actual equation to the curve-locus, divide (8) by (7),

which gives $$\frac{4a^2-4a^2}{3a^2+a^2} = \frac{y}{x} = \lambda, \text{ say,}$$

whence $$a^2 = a^2\,\frac{4-3\lambda}{4+\lambda},$$

$$3a^2+a^2 = a^2\,\frac{16}{4+\lambda}, \quad 5a^2-a^2 = a^2\,\frac{8\,(2+\lambda)}{4+\lambda}.$$

Hence, squaring (7), substituting from the above, and restoring the value of λ, we have $$(4x-3y)(2x+y)^2 = 4a^2\,(4x+y),$$

which shows that the locus sought is a nodal cubic curve, of which the asymptotes are given by $(4x-3y)(2x+y)^2 = 0$, or, $4x = 3y$, and $2x+y = 0$, and which clearly passes through each of the points A, O, B.

Again, differentiating (7) and (8), we have the equations

$$\frac{1}{a^2}\,\frac{dx}{da} = \frac{a^4+14a^2a^2-15a^4}{a^2\,(5a^2-a^2)^2} \qquad \text{.....(9),}$$

$$\frac{1}{a^2}\,\frac{dy}{da} = -\,\frac{4a^4+8a^2a^2+20a^4}{a^2\,(5a^2-a^2)^2} \qquad \text{.....(10).}$$

Hence, if S be the area of the curve, we have

$$dS = y\,dx = \frac{4a^2}{a}\cdot\frac{a^2-a^2}{5a^2-a^2}\cdot\frac{a^2\,(a^4+14a^2a^2-15a^4)}{a^2\,(5a^2-a^2)^2}\,da$$

$$= 4a^4\,\frac{(a^2-a^2)(a^4+14a^2a^2-15a^4)}{a^3\,(5a^2-a^2)^4}\,da.$$

The necessary integrations may be performed after a previous application of the theory of partial fractions; but it is more commodious to use the trigonometrical transformation $a^2 = 5a^2\sin^2\phi$, leading to

$$a = 5^{\frac{1}{2}}\,a\sin\phi, \quad da = 5^{\frac{1}{2}}\,a\cos\phi\,d\phi.$$

The transformed equation is

$$dS = \frac{4a^2}{125}\cdot\frac{(1-5\sin^2\phi)(5\sin^4\phi+14\sin^2\phi-3)}{\sin^3\phi\cos^5\phi}\,d\phi,$$

whence we have, immediately,

$$\frac{125}{4a^2}\,S = -25\int\frac{\sin^3\phi}{\cos^5\phi}\,d\phi - 65\int\frac{\sin\phi}{\cos^5\phi}\,d\phi + 29\int\frac{d\phi}{\sin\phi\cos^5\phi} - 3\int\frac{d\phi}{\sin^3\phi\cos^5\phi};$$

and, since it is known that

$$\int\frac{\sin^3\phi}{\cos^5\phi}\,d\phi = \tfrac{1}{4}\tan^4\phi, \quad \int\frac{\sin\phi}{\cos^5\phi}\,d\phi = \tfrac{1}{4}\sec^4\phi,$$

$$\int\frac{d\phi}{\sin\phi\cos^5\phi} = \tfrac{1}{4}\sec^4\phi + \tfrac{1}{2}\sec^2\phi + \log\tan\phi,$$

$$\int\frac{d\phi}{\sin^3\phi\cos^5\phi} = \frac{1}{4\sin^2\phi\cos^4\phi} - 3\,\frac{\cos 2\phi}{\sin^2 2\phi} + 3\log\tan\phi,$$

the value of S for any limits of ϕ can be easily calculated. Similarly, if it were required to calculate the length (L) of any portion of the perimeter of the cubic curve, we should have

$$\left(\frac{1}{a^2}\cdot\frac{dL}{da}\right)^2 = \left(\frac{1}{a^2}\cdot\frac{dx}{da}\right)^2 + \left(\frac{1}{a^2}\cdot\frac{dy}{da}\right)^2$$

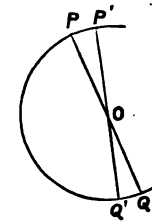
$$= \frac{(a^4+14a^2a^2-15a^4)^2+(4a^4+8a^2a^2+20a^4)^2}{a^4\,(5a^2-a^2)^4},$$

whence $\frac{L}{a^2} = \int \frac{\{17a^8 + 92a^2a^6 + 390a^4a^4 - 100a^6a^2 + 625a^8\}^{\frac{1}{2}}}{a^2(5a^2 - a^2)^2}\, da,$

which is a hopelessly complicated hyper-elliptic form.

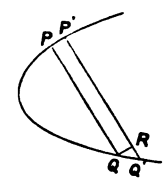
4035. (By Professor WOLSTENHOLME, M.A., Sc.D.)—If, in any curve, PQ be a chord normal at P, then, when PQ is of maximum or minimum length, either PQ is normal also at Q, or Q is the centre of curvature at P.

Solution.



Let PQ be the normal at P, and let its length be a maximum or a minimum; then, draw a normal to the curve at the consecutive point P′, and let O be the point of intersection of the two normals. Since PQ is a maximum or a minimum, we have $d(PQ) = 0$, or $PQ = P'Q'$, and, as usual, the lines are bisected at O; whence, by elementary geometry, the triangles POP′, QOQ′ are equal, and therefore the angle at Q, which is equal to the angle at P, is a right angle, and PQ is normal to the curve also at Q. In the particular case when P′Q′ intersects PQ at Q, that is, when the point of intersection O coincides with Q, Q becomes the point of intersection of two consecutive normals, and is accordingly the centre of curvature at P.

[The following *Solution* by ABPARAGUS is inserted in correction of some errors that were discovered in the foregoing solution after it had been set in type:—



Let PQ, P′Q′ be consecutive chords, normal at P, P′ respectively, and draw QR perpendicular to P′Q′, then $P'R = PQ$, or $d.PQ = RQ'$; hence, when PQ is of maximum or minimum length, $RQ' = 0$. This can only happen in two ways (if the curve have no singular point at Q): (1) the angle QQ′R may be a right angle, or PQ normal at Q also; or (2) R may coincide with Q, in which case Q is the centre of curvature at P.

An exception should be made for singular points; thus, the axis of a cardioid is a minimum normal chord, though not satisfying either of the stated conditions.]

4167. (By the EDITOR.)—Prove that the sum to infinity of the series

$$\frac{x}{1!} + \frac{x^4}{4!} + \frac{x^7}{7!} + \ldots + \frac{x^{3n-2}}{(3n-2)!} + \ldots \text{ ad inf.}$$

is $$\tfrac{1}{3}\left\{e^x + \frac{2}{e^{\frac{1}{2}x}} \sin \tfrac{1}{2}(x\sqrt{3} - \tfrac{1}{3}\pi)\right\},$$

and show what the series becomes when $x = \frac{1}{6}\pi\sqrt{3}$.

Solution.

Let $$y = \frac{x}{1!} + \frac{x^4}{4!} + \frac{x^7}{7!} + \ldots,$$

therefore $$\frac{dy}{dx} = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \ldots, \quad \frac{d^2y}{dx^2} = \frac{x^2}{2!} + \frac{x^5}{5!} + \ldots,$$

whence $$\frac{d^3y}{dx^3} + \frac{dy}{dx} + y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = e^x.$$

Differentiating, $$\frac{d^4y}{dx^4} + \frac{d^3y}{dx^3} + \frac{dy}{dx} = e^x = \frac{d^2y}{dx^2} + \frac{dy}{dx} + y,$$

or, $$\frac{d^3y}{dx^3} - y = 0.$$

Assuming $y = Ce^{mx}$, we get the auxiliary equation $m^3 - 1 = 0$, the roots of which are $m = 1$, $m = -\frac{1}{2} + \frac{1}{2}\sqrt{3}\,i$, $m = -\frac{1}{2} - \frac{1}{2}\sqrt{3}\,i$, where, as usual, $i^2 + 1 = 0$. Therefore, the complete primitive is

$$y = Ce^x + e^{-\frac{1}{2}x}(A\cos\tfrac{1}{2}\sqrt{3}\,x + B\sin\tfrac{1}{2}\sqrt{3}\,x),$$

which gives

$$\frac{dy}{dx} = Ce^x - \tfrac{1}{2}e^{-\frac{1}{2}x}(A\cos\tfrac{1}{2}\sqrt{3}\,x + B\sin\tfrac{1}{2}\sqrt{3}\,x) + \tfrac{1}{2}\sqrt{3}\,e^{-\frac{1}{2}x}(-A\sin\tfrac{1}{2}\sqrt{3}\,x + B\cos\tfrac{1}{2}\sqrt{3}\,x),$$

$$\frac{d^2y}{dx^2} = Ce^x - \tfrac{1}{2}e^{-\frac{1}{2}x}(A\cos\tfrac{1}{2}\sqrt{3}\,x + B\sin\tfrac{1}{2}\sqrt{3}\,x) - \tfrac{1}{2}\sqrt{3}\,e^{-\frac{1}{2}x}(-A\sin\tfrac{1}{2}\sqrt{3}\,x + B\cos\tfrac{1}{2}\sqrt{3}\,x).$$

Now, when $x = 0$, we must have

$$y = 0, \quad \frac{dy}{dx} = 1, \quad \frac{d^2y}{dx^2} = 0.$$

Therefore $0 = C + A$, $1 = C - \frac{1}{2}A + \frac{1}{2}\sqrt{3}\,B$, $0 = C - \frac{1}{2}A - \frac{1}{2}\sqrt{3}\,B$,

whence $$A = -\tfrac{1}{3}, \quad B = \frac{1}{\sqrt{3}}, \quad C = \tfrac{1}{3}.$$

Therefore, finally, we have

$$y = \tfrac{1}{3}e^x + e^{-\frac{1}{2}x}\left(-\tfrac{1}{3}\cos\tfrac{1}{2}\sqrt{3}\,x + \frac{1}{\sqrt{3}}\sin\tfrac{1}{2}\sqrt{3}\,x\right)$$

$$= \tfrac{1}{3}e^x + \tfrac{1}{3}e^{-\frac{1}{2}x}(\sqrt{3}\sin\tfrac{1}{2}\sqrt{3}\,x - \cos\tfrac{1}{2}\sqrt{3}\,x).$$

Let $$\cot\theta = \sqrt{3}, \quad \theta = \tfrac{1}{6}\pi,$$

and $$y = \tfrac{1}{3}e_x + \tfrac{1}{3}e^{-\frac{1}{2}x}\left\{\cot\theta\sin\tfrac{1}{2}\sqrt{3}\,x - \cos\tfrac{1}{2}\sqrt{3}\,x\right\}$$

$$= \tfrac{1}{3}e^x + \tfrac{1}{3}e^{-\frac{1}{2}x}\frac{\sin(\frac{1}{2}\sqrt{3}\,x - \theta)}{\sin\theta} = \tfrac{1}{3}e^x + \tfrac{2}{3}e^{-\frac{1}{2}x}\sin(\tfrac{1}{2}\sqrt{3}\,x - \tfrac{1}{6}\pi).$$

Putting $x = \frac{\pi}{3\sqrt{3}}$, we have the interesting identity

$$\frac{\pi}{3\sqrt{3}} + \left(\frac{\pi}{3\sqrt{3}}\right)^4\frac{1}{4!} + \ldots = \tfrac{1}{3}e^{\pi/3^{\frac{3}{2}}}.$$

4333. (By R. Tucker, M.A.)—Solve the equations

$$x^2+y^2+z^2-xy+yz-xz = a = 15,$$
$$x^2+y^2+z^2-xy-yz+xz = b = 9,$$
$$x^2+y^2+z^2+xy-yz-xz = c = 7.$$

Solution.

Writing $x^2+y^2+z^2 = \lambda$ ..... (1),

we have the system $-xy+yz-zx = a-\lambda$ ..... (2),

$$-xy-yz+zx = b-\lambda, \quad +xy-yz-zx = c-\lambda \text{ ..... (3, 4).}$$

Adding (2), (3), (4), we get $-xy-yz-zx = a+b+c-3\lambda$ ..... (5).

Subtracting (5) from (2), (3), (4), successively, we have

$$yz = \lambda-\tfrac{1}{2}(b+c) = \lambda-\alpha, \quad zx = \lambda-\tfrac{1}{2}(c+a) = \lambda-\beta \text{ ..... (6, 7),}$$
$$xy = \lambda-\tfrac{1}{2}(a+b) = \lambda-\gamma \text{ ..... (8),}$$

which, when multiplied together, lead to

$$x^2y^2z^2 = (\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma) \text{ ..... (9).}$$

Dividing (9) by the squares of (6), (7), (8), in turn, we get

$$x^2 = \frac{(\lambda-\beta)(\lambda-\gamma)}{\lambda-\alpha}, \quad y^2 = \frac{(\lambda-\gamma)(\lambda-\alpha)}{\lambda-\beta} \text{ ..... (10, 11),}$$

$$z^2 = \frac{(\lambda-\alpha)(\lambda-\beta)}{\lambda-\gamma} \text{ ..... (12).}$$

Moreover, substituting from (10), (11), (12) in (1), we get

$$(\lambda-\alpha)^2(\lambda-\beta)^2+(\lambda-\beta)^2(\lambda-\gamma)^2+(\lambda-\gamma)^2(\lambda-\alpha)^2 = \lambda(\lambda-\alpha)(\lambda-\beta)(\lambda-\gamma) \text{ ..... (13),}$$

which is a quartic in λ; and, when λ is determined from (13) in terms of the known quantities α, β, γ, we get the value of x, y, z immediately from (10), (11), (12).

The biquadratic in λ may be written

$$2\lambda^4-3(\alpha+\beta+\gamma)\lambda^3+\{2(\alpha^2+\beta^2+\gamma^2)+3(\alpha\beta+\beta\gamma+\gamma\alpha)\}\lambda^2$$
$$-2\{\alpha\beta(\alpha+\beta)+\beta\gamma(\beta+\gamma)+\gamma\alpha(\gamma+\alpha)-\tfrac{1}{2}\alpha\beta\gamma\}\lambda+\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2=0 \text{ ...(14).}$$

Hence, if we take the particular case $a = 15$, $b = 9$, $c = 7$, we have $\alpha = 8$, $\beta = 11$, $\gamma = 12$, and (14) gives for the quartic equation in λ

$$2\lambda^4-93\lambda^3+1606\lambda^2-12200\lambda+34384 = 0,$$

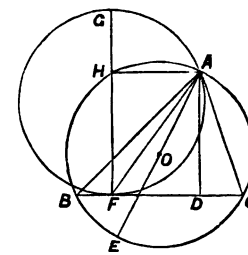
and, since this may be written $(\lambda-14)(2\lambda^3-65\lambda^2+696\lambda-2456) = 0$,

we have, for one root, $\lambda = 14$, which gives $x^2 = 1$, $y^2 = 4$, $z^2 = 9$. The other values of λ are given by the cubic $2\lambda^3-65\lambda^2+696\lambda-2456 = 0$,

an equation having only one real root which lies between 8 and 9, and which can easily be calculated by Horner's Method.

4434. (By the Editor.)—Given the base and vertical angle, construct a triangle such that the rectangle under the sides shall have a given ratio to the square of a line drawn from the vertex to a fixed point in the base.

Solution.



Let BC be the given base; on BC describe a segment of a circle ABC, containing the given angle; then the locus of the vertex is this circular arc. Complete the circle, and take O to be its centre. Let F be the fixed point in the base; draw FH at right angles to BC, and measure on it a length FG, such that FG may be to the diameter of the circle as the rectangle under the sides is to the square of the line drawn from the vertex to the fixed point F; so that, if n be the given ratio, we have FG = n times the diameter. On FG as diameter, describe a circle cutting the circle BAC in A. Join AB, AC, AF; then ABC shall be the triangle required. Draw AD perpendicular to BC, AH perpendicular to FG, and AOE the diameter of the circle through A. Then we have $AF^2 = GF \cdot FH = n \cdot AE \cdot AD = n \cdot AB \cdot AC$, which shows that the triangle ABC satisfies the demands of the problem, and is accordingly the triangle required. Since the circle on FG as diameter intersects the circle ABC in two points, there are, *in general*, two triangles possessing the properties sought.

4552. (By Professor CROFTON, F.R.S.)—If x be a first approximation to the n^{th} root of a number N, prove (1) that the root will be

$$\frac{(n+1)\,N + (n-1)\,x^n}{(n-1)\,N + (n+1)\,x^n}\,x, \text{ nearly,}$$

and find (2) whether this is a better approximation than NEWTON'S, viz.,

$$\frac{N + (n-1)\,x^n}{nx^{n-1}}.$$

Solution.

1. Since x is a near approximation to the n^{th} root of N, we may take $N^{1/n} = x + \epsilon$, where ϵ is small; so that $N = (x+\epsilon)^n = x^n + nx^{n-1}\epsilon$, neglecting the second and higher power of ϵ. Hence we obtain

$$\epsilon = \frac{N - x^n}{nx^{n-1}}, \text{ and } x + \epsilon = \frac{N + (n-1)\,x^n}{nx^{n-1}},$$

which is NEWTON'S approximation. If, however, we retain the second power of ϵ, we have

$$\begin{aligned} N = (x+\epsilon)^n &= x^n + nx^{n-1}.\epsilon + \frac{n(n-1)}{1.2}\,x^{n-2}\,\epsilon^2 \\ &= x^n + \epsilon\left\{nx^{n-1} + \tfrac{1}{2}n(n-1)\,x^{n-2}.\epsilon\right\} \\ &= x^n + \epsilon\left\{nx^{n-1} + \tfrac{1}{2}n(n-1)\,x^{n-2}.\frac{N-x^n}{nx^{n-1}}\right\} \\ &= x^n + \epsilon.\frac{N(n-1) + (n+1)\,x^n}{2x}, \end{aligned}$$

whence, easily, $$x+\epsilon = \frac{(n+1)\,N+(n-1)\,x^n}{(n-1)\,N+(n+1)\,x^n}\,x,$$

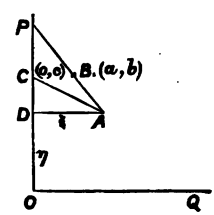
which is the formula in question, and which appears to have been first given by HUTTON (*Mathematical Tracts*, Vol. I. (1812), p. 212, Para. 5.)

2. From the above demonstration, it is clear that HUTTON's approximation must be closer than that of NEWTON.

4620. (By the EDITOR.)—Two straight lines at right angles to one another being given in position, it is required to find in one of them a point, such that its distance from the other, measured along a line drawn through itself and a given point, shall have a given ratio to its distance from a given point in the other line.

Solution.

Let us consider the more general question of determining the locus of a point A (ξ, η), such that its distance from OP, measured along AB, which passes through the given point B (a, b), shall have a given ratio k to AC, which joins A to the given point C $(0, c)$.



Hence, since the equation of AB is

$$\frac{x-\xi}{y-\eta} = \frac{\xi-a}{\eta-b},$$

we have the intercept $$OP = \frac{b\xi-a\eta}{\xi-a},$$

and $$DP = OP-OD = \frac{b\xi-a\eta}{\xi-a} - \eta = \frac{(b-\eta)\,\xi}{\xi-a},$$

whence $$AP^2 = AD^2+DP^2 = \frac{\xi^2}{(\xi-a)^2}\left\{(\xi-a)^2+(\eta-b)^2\right\}.$$

Therefore, since $AC^2 = \xi^2+(\eta-c)^2$, and $AP^2 = k^2 . AC^2$, we get

$$\xi^2\left\{(\xi-a)^2+(\eta-b)^2\right\} = k^2\,(\xi-a)^2\left\{\xi^2+(\eta-c)^2\right\},$$

which may be written

$$(1-k^2)\,x^2\,(x-a)^2 = k^2\,(x-a)^2\,(y-c)^2-x^2\,(y-b)^2 \quad\ldots\ldots\ldots\ldots\ldots(1),$$

showing that the locus required is a quartic curve. If we require to find, on the axis of x, points satisfying the condition of the problem, we have to seek the intersection of (1) with $y = 0$, viz., the four values of x are given by $$(1-k^2)\,x^2\,(x-a)^2 = k^2c^2\,(x-a)^2-b^2x^2.$$

A reduction is effected when the ratio is one of equality, viz., if $k=1$, the quartic reduces to $$(x-a)^2\,(y-c)^2-x^2\,(y-b)^2 = 0,$$

or $$\left\{(x-a)\,(y-c)+x\,(y-b)\right\}\left\{(x-a)\,(y-c)-x\,(y-b)\right\} = 0,$$

one of which is the hyperbola $xy = \frac{1}{2}(b+c)x + \frac{1}{2}ay - \frac{1}{2}ac$,

and the other the right cone $y = \frac{b-c}{a}x + c$.

The particular points on the axis of x are determined by its intersection with these, viz., there are, in this case, only two points given by

$$x = \frac{ac}{c \pm b}.$$

5430. (By the Editor.)—Find the average length of the radius of curvature of a parabola, for points situated between the vertex and the latus-rectum.

Solution.

If the parabola be $y^2 = 4ax$, we have $\frac{dy}{dx} = \left(\frac{a}{x}\right)^{\frac{1}{2}}$, $\frac{d^2y}{dx^2} = -\frac{1}{2}\left(\frac{a}{x^3}\right)^{\frac{1}{2}}$,

so that the radius of curvature is given by $\rho = \frac{2}{\sqrt{a}}(x+a)^{\frac{3}{2}}$,

and $$\int \rho\, dx = \frac{2}{\sqrt{a}}\int (x+a)^{\frac{3}{2}}\, dx = \frac{4}{5\sqrt{a}}(x+a)^{\frac{5}{2}},$$

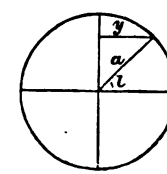
whence $$\int_0^a \rho\, dx = \frac{4(4\sqrt{2}-1)}{5}a^2.$$

Therefore $$M = \left(\int_0^a \rho\, dx\right) \Big/ \left(\int_0^a dx\right) = \frac{4(4\sqrt{2}-1)}{5}a = ka, \text{ say,}$$

where $k = 3{\cdot}72548352$ approximately.

5431. (By the Editor.)—Supposing the Hugli river to flow due South, prove that the pressure on the Western bank at the depth d would be increased by the change of latitude of the running water in the ratio $(gd + bv\omega \sin l) : gd$, where b is the breadth of the river, v its velocity, l the latitude, and ω the angular velocity of the earth about its axis.

Solution.



If y be the radius of the circle described by a point in latitude l, we have $y = a\cos l$; and, the time of describing δs being $\delta s / v$, and the change of velocity $\delta(a\omega\cos l) = -a\omega \sin l\, \delta l$, the acceleration eastward is

$$av\omega \sin l \frac{dl}{ds} = v\omega \sin l,$$

since $dl = ads$; therefore, the additional pressure on the Western bank is $\rho bv\omega \sin l$, and the ratio required is $(gd + bv\omega \sin l) : gd$.

5434. (By the EDITOR.)—Find the volume between $x = 0$ and $x = 2l$ of the solid bounded by the surface whose equation is

$$a(y^4 + z^4) - z^2(x^2 - 2ay^2 + 2c^2) - y^2(bx^2 + c^2x + c^3) = 0.$$

Solution.

Since the equation of the surface may obviously be put into the form

$$a(y^2 + z^2)^2 - z^2(x^2 + 2c^2) - y^2(bx^2 + c^2x + c^3) = 0,$$

we have, by putting $y = r\cos\theta$, $z = r\sin\theta$, and regarding x as constant,

$$ar^2 = (x^2 + 2c^2)\sin^2\theta + (bx^2 + c^2x + c^3)\cos^2\theta,$$

which is the polar equation of the curve of intersection of the surface with a plane parallel to the coordinate plane of yz. If A be the area of this curve, we get

$$A = 4 \cdot \tfrac{1}{2}\int_0^{\frac{1}{2}\pi} r^2\,d\theta$$

$$= \frac{2}{a}(x^2 + 2c^2)\int_0^{\frac{1}{2}\pi}\sin^2\theta\,d\theta + \frac{2}{a}(bx^2 + c^2x + c^3)\int_0^{\frac{1}{2}\pi}\cos^2\theta\,d\theta$$

$$= \frac{\pi}{2a}(x^2 + bx^2 + c^2x + 3c^3).$$

Therefore, if V be the volume required, we have

$$V = \int_0^{2l} A\,dx = \frac{\pi}{2a}\int_0^{2l}(x^2 + bx^2 + c^2x + 3c^3)\,dx$$

$$= \frac{\pi}{2a}\left[\tfrac{1}{3}x^4 + \tfrac{1}{3}bx^3 + \tfrac{1}{2}c^2x^2 + 3c^3x\right]_0^{2l} = \frac{\pi l}{3a}(6l^2 + 4bl^2 + 3c^2l + 9c^3).$$

5798. (By Professor COCHEZ.)—Construire un triangle rectangle, connaissant les sommes de chacun des côtés avec le perpendiculaire abaissée du sommet de l'angle droit sur l'hypotenuse.

6263. (By J. YOUNG, B.A.)—Show that the solution of Quest. 5798, given on p. 56 of Vol. XXXI., is incorrect.

Solution.

Let a, b be the sides of the right-angled triangle, and x the perpendicular from the right angle on the hypotenuse. Then $a + x = m$, $b + x = n$, m and n being constants. Again, from elementary geometry, we know that

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{x^2},$$

whence

$$\frac{1}{(m-x)^2} + \frac{1}{(n-x)^2} = \frac{1}{x^2},$$

which equation, in its developed form, is equivalent to the biquadratic

$$x^4 - 4mn \cdot x^2 + 2mn(m+n)x - m^2n^2 = 0.$$

Identifying this with the standard form

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0,$$

we have $a = 1$, $b = 0$, $c = -\frac{1}{2}mn$, $d = \frac{1}{4}mn(m+n)$, $e = -m^2n^2$,

so that $I = \frac{1}{3}m^2n^2$, $G = \frac{1}{3}mn(m+n)$,

$$J = \frac{m^2n^2}{108}(50mn - 27m^2 - 27n^2),$$

and the reducing cubic is $4\theta^3 - I\theta + J = 0$;

which gives $\theta = \sqrt[3]{p} + \sqrt[3]{q}$,

where $p = 8\left\{-J + (J^2 - \frac{1}{27}I^3)^{\frac{1}{2}}\right\}$, $q = 8\left\{-J - (J^2 - \frac{1}{27}I^3)^{\frac{1}{2}}\right\}$,

and, accordingly, x is given by

$$x = A + B - \frac{1}{3}\frac{G}{AB},$$

where $A^3 = \frac{1}{8}(2mn + \theta_1)$, $B^3 = \frac{1}{8}(2mn + \theta_2)$.

By actual calculation, we find

$$J^2 - \tfrac{1}{27}I^3 = \frac{m^4n^4}{432}(m-n)^2(27m^2 - 46mn + 27n^2),$$

so that $(J^2 - \frac{1}{27}I^3)^{\frac{1}{2}} = \frac{m^2n^2(m-n)}{12\sqrt{3}}(27m^2 - 46mn + 27n^2)^{\frac{1}{2}}$.

Lastly, as x is known, we know the sides a, b from $a = m - x$, $b = n - x$, and therefore the triangle can be constructed.

5873. (By the Editor.)—Prove that (1) the locus of the poles, with regard to the auxiliary circle, of tangents to the corresponding ellipse, is a similar ellipse with its minor axis equal to the major axis of the first ellipse; and (2) the locus of points taken in the same manner with regard to the latter ellipse is another ellipse similar to the first.

Solution.

Let the given ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .....(1),

so that any tangent is $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi = 1$ .....(2),

and the auxiliary circle is $x^2 + y^2 = a^2$... .....(3).

If (ξ, η) be the pole of (2) with regard to (3), it is clear that (2) must be identical with $\xi x + \eta y = a^2$, which is the polar of (ξ, η) with respect to (3).

Hence, $\frac{\cos\phi}{a} = \frac{\xi}{a^2}$, $\frac{\sin\phi}{b} = \frac{\eta}{a^2}$,

which, by the elimination of ϕ, give

$$\frac{\xi^2}{a^2} + \frac{b^2\eta^2}{a^4} = 1, \text{ or } \frac{\xi^2}{\alpha^2} + \frac{\eta^2}{\beta^2} = 1 \text{ .....(4)},$$

where $\alpha = a$, $\beta = a^2/b$; and, since $a > b$, we have $(a^2/b) > a$, or $\beta > \alpha$, so that α is the semi-axis-minor of the locus of (ξ, η), and is equal to the semi-axis-major of the primitive ellipse.

Again, the auxiliary circle of (4) is $x^2+y^2=\beta^2$, and any tangent is

$$\frac{x}{\alpha}\cos\theta+\frac{y}{\beta}\sin\theta=1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5),$$

so that, if X, Y be the pole of (5) with regard to $x^2+y^2=\beta^2$, we have

$$\frac{\cos\theta}{\alpha}=\frac{X}{\beta^2}, \quad \frac{\sin\theta}{\beta}=\frac{Y}{\beta^2},$$

whence

$$\frac{\alpha^2X^2}{\beta^4}+\frac{Y^2}{\beta^2}=1,$$

which reduces to

$$\frac{X^2}{a^2}+\frac{Y^2}{b^2}=\frac{a^4}{b^4}.$$

5949. (By J. Young, B.A.)—If the equation of a conic referred to any tangent and normal as axes, which is of the form

$$ax^2+2hxy+by^2+2fy=0,$$

is transformed to the tangent and normal at any other point on the curve, prove that, in addition to the usual invariants $a+b$, $ab-h^2$, there is in this case a third invariant, viz., af^2.

Solution.

First, consider the most general transformation possible of the conic

$$S\equiv ax^2+2hxy+by^2+2gx+2fy+c=0,$$

viz., transforming the equation to parallel axes through (x', y'), it becomes

$$ax^2+2hxy+by^2+2\gamma x+2\phi y+\sigma=0,$$

where

$$\gamma=ax'+hy'+g, \quad \phi=hx'+by'+f,$$

$$\sigma=ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=\gamma x'+\phi y'+(gx'+fy'+c).$$

Next, turning the axes through an angle θ, this reduces to

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0,$$

where

$$A=a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta,$$

$$H=h\cos2\theta+(b-a)\sin\theta\cos\theta, \quad B=a\sin^2\theta-2h\sin\theta\cos\theta+b\cos^2\theta,$$

$$G=\gamma\cos\theta+\phi\sin\theta, \quad F=-\gamma\sin\theta+\phi\cos\theta, \quad C=\sigma.$$

Now, if the conic be initially referred to any tangent and normal, we have $g=0$, $c=0$, and, if the transformation be also to the tangent and normal at any other point on the curve, we have $G=0$, $C=\sigma=0$. Hence, since $G=0$ gives

$$\tan\theta=\frac{-\gamma}{\phi}, \quad \sin\theta=\frac{-\gamma}{(\gamma^2+\phi^2)^{\frac{1}{2}}}, \quad \cos\theta=\frac{\phi}{(\gamma^2+\phi^2)^{\frac{1}{2}}},$$

we have

$$F=\frac{\gamma^2}{(\gamma^2+\phi^2)^{\frac{1}{2}}}+\frac{\phi^2}{(\gamma^2+\phi^2)^{\frac{1}{2}}}, \quad \text{or} \quad F^2=\gamma^2+\phi^2.$$

Again,

$$A=\frac{a\phi^2-2h\phi\gamma+b\gamma^2}{\gamma^2+\phi^2},$$

so that $AF^2 = a\phi^2 - 2h\phi\gamma + b\gamma^2 = \phi(a\phi - h\gamma) - \gamma(h\phi - b\gamma)$

$= \phi\{(af - hg) - (h^2 - ab)y'\} - \gamma\{(hf - bg) + (h^2 - ab)x'\}$

$= f(a\phi - h\gamma) - g(h\phi - b\gamma) - (h^2 - ab)(\gamma x' + \phi y')$

$= f\{(af - hg) - (h^2 - ab)y'\} - g(h\phi - b\gamma) - (h^2 - ab)(\gamma x' + \phi y')$

$= f(af - hg) - g(h\phi - b\gamma) - (h^2 - ab)\{\gamma x' + (\phi + f)y'\}$.

Now, when $g = 0$, $c = 0$, $\sigma = 0$, we have

$$\sigma = \gamma x' + (\phi + f)y' = 0.$$

Hence, finally, $AF^2 = af^2$, which shows that af^2 is an invariant.

[The PROPOSER remarks that "the property here enunciated was noticed by me when solving a question in curvature; my proof was similar to that now given, but there is, I believe, a simple geometrical interpretation. Since contributing the question, I have noticed that Dr. SALMON states (without proof) that the discriminant $abc + nfgh - af^2 - bg^2 - ch^2$ is an invariant, and for the equation of the question this expression becomes $-af^2$."]

6069. (By the EDITOR.)—Find the locus of the centres of circles tangential to a parabola and the tangent at its vertex.

Solution.

The equation to the parabola, referred to its principal axis and the tangent at the vertex, is $y^2 = 4ax$. If (ξ, η) be the coordinates, in any position, of the circle which touches both the parabola and the tangent, it is clear that the length of the radius of the circle is ξ, so that its equation may be written $(x - \xi)^2 + (y - \eta)^2 = \xi^2$ .....(1).

If, therefore, (α, β) be the point of contact of the circle and the parabola, we have $(\alpha - \xi)^2 + (\beta - \eta)^2 = \xi^2$ .....(2),

and $\beta^2 = 4a\alpha$ .....(3).

Again, since the curves touch each other at (α, β), they have a common tangent at the point of contact, and the value of dy/dx for both the curves, when $x = \alpha$, $y = \beta$, must be the same. Now, the value of dy/dx for the circle and for the parabola is easily found to be

$$\frac{dy}{dx} = -\frac{x - \xi}{y - \eta}, \quad \frac{dy}{dx} = \frac{2a}{y},$$

respectively. Hence, we have the relation

$$\frac{2a}{\beta} = -\frac{\alpha - \xi}{\beta - \eta},$$

which gives $\beta(2a + \alpha - \xi) = 2a\eta$ .....(4).

Substituting in (2) and (4) the value of α in terms of β from (3), we have

$$\left(\xi - \frac{\beta^2}{4a}\right)^2 + (\eta - \beta)^2 = \xi^2, \quad \beta\left(2a + \frac{\beta^2}{4a} - \xi\right) = 2a\eta,$$

which may be reduced to the forms

$$\beta^4 + (16a^2 - 8a\xi)\beta^2 - 32a^2\eta \cdot \beta + 16a^2\eta^2 = 0 \quad \text{.....(5)},$$

$$\beta^3 + (8a^2 - 4a\xi)\beta - 8a^2\eta = 0 \text{.....(6)}.$$

If we now eliminate β between these equations, we obtain the locus sought. For this purpose, multiplying (6) by β, we get

$$\beta^4 + (8a^2 - 4a\xi)\,\beta^2 - 8a^2\eta \,.\, \beta = 0,$$

which, when subtracted from (5), gives

$$(8a^2 - 4a\xi)\,\beta^2 - 24a^2\eta \,.\, \beta + 16a^2\eta^2 = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (7).$$

Multiply (6) by $(8a^2 - 4a\xi)$, and (7) by β; from the former subtracting the latter, and making some obvious reductions, we get

$$3\eta \,.\, \beta^2 + 2\left\{(2a-\xi)^2 - \eta^2\right\}\beta - 4a\eta\,(2a-\xi) = 0 \quad \ldots\ldots\ldots\ldots (8).$$

Dividing (7) by $4a$, we have $(2a-\xi)\,\beta^2 - 6a\eta \,.\, \beta + 4a\eta^2 = 0 \ldots\ldots\ldots\ldots\ldots (9)$.

We can now write these in the forms

$$\left.\begin{aligned} A\,\beta^2 + 2B\,\beta + C &= 0 \\ A_1\beta^2 + 2B_1\beta + C_1 &= 0 \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

where $A = 3\eta$, $B = (2a-\xi)^2 - \eta^2$, $C = -4a\eta\,(2a-\xi)$,

$A_1 = 2a-\xi$, $B_1 = -3a\eta$, $C_1 = 4a\eta^2$,

so that, by actual calculation, we have the invariants

$$I = AC - B^2 = -\left\{(2a-\xi)^4 + \eta^4 + 2\eta^2\,(2a-\xi)(4a+\xi)\right\},$$

$$I' = A_1C_1 - B_1^2 = -a\eta^2\,(a+4\xi),$$

$$J = AC_1 + A_1C - 2BB_1 = 2a\eta\left\{3\eta^2 + (2a-\xi)^2\right\}.$$

Now, it is well known that the eliminant of the system given by (10) is $J^2 = 4II'$, which at once leads to the equation

$$4a\eta^2\,(a+4\xi)\left\{(2a-\xi)^4 + \eta^4 + 2\eta^2\,(2a-\xi)(4a+\xi)\right\} = 4a^2\eta^2\left\{3\eta^2 + (2a-\xi)^2\right\}^2,$$

which, on expansion and reduction, gives for the locus sought the quartic curve

$$\eta^2\left\{2\xi^2 + 10a\xi - a^2\right\} = 2\xi\left\{\xi^3 - 3a\xi^2 + 6a^2\xi - 4a^3\right\},$$

which may also be written in the form

$$2x^4 - 6ax^3 + 2\,(6a^2 - y^2)\,x^2 - 2a\,(4a^2 + 5y^2)\,x + a^2y^2 = 0.$$

5964 & 6084. (By J. J. Walker, M.A., F.R.S.)—If P be any point on a plane curve, O the pole, ON the line from which θ is measured, and PN the normal to the curve at P, prove that $ON = (r\,dr) / d(r\cos\theta)$.

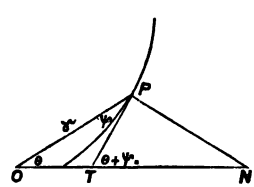
Solution.

From the geometry of the figure, we have

$$\frac{OP}{ON} = \frac{\cos(\theta+\psi)}{\cos\psi} = \cos\theta - \sin\theta\tan\psi$$

$$= \cos\theta - \sin\theta \,.\, \frac{r\,d\theta}{dr} = \frac{d(r\cos\theta)}{dr}.$$

Therefore $$ON = \frac{r\,dr}{d(r\cos\theta)}.$$



[For another solution, see Vol. xxxii., pp. 58–59.]

6127. (By Professor BRAHY.)—Si l'on coupe un hyperboloïde à une nappe par un plan et que, par tous les points de la courbe d'intersection, on mène des plans tangents à l'hyperboloïde, quelle sera la surface enveloppée des plans tangents.

Solution.

It is geometrically evident that the envelope required is the tangent cone to the quadric, passing through its curve of intersection with the given plane; so that, if the hyperboloid be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad \text{.....(1),}$$

and the plane be $lx + my + nz = p$ .....(2),

the equation to the envelope is written down at sight, viz., it is the homogeneous equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \left(\frac{lx + my + nz}{p}\right)^2 \quad \text{.....(3).}$$

The same result follows with ease analytically, for, if (x_1, y_1, z_1) be any point on the surface, we have to seek the envelope of

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - \frac{zz_1}{c^2} = 1 \quad \text{.....(4),}$$

when $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{z_1^2}{c^2} = 1, \quad lx_1 + my_1 + nz_1 = p$ .....(5, 6).

Differentiating (4), (5), (6),

$$\frac{x}{a^2}dx_1 + \frac{y}{b^2}dy_1 - \frac{z}{c^2}dz_1 = 0, \quad \frac{x_1}{a^2}dx_1 + \frac{y_1}{b^2}dy_1 - \frac{z_1}{c^2}dz_1 = 0,$$

$$ldx_1 + mdy_1 + ndz_1 = 0,$$

and employing the arbitrary multipliers λ, μ, we have

$$\frac{x}{a^2} = \lambda \cdot \frac{x_1}{a^2} = \mu l, \quad \frac{y}{b^2} = \lambda \cdot \frac{y_1}{b^2} = \mu m, \quad \frac{z}{c^2} = \lambda \cdot \frac{z_1}{c^2} = -\mu n,$$

whence $x = \lambda x_1$, $y = \lambda y_1$, $z = \lambda z_1$, which, when substituted in (5) and (6) immediately lead, by the elimination of λ, to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \frac{(lx + my + nz)^2}{p^2},$$

the same cone as before.

6903. (By W. J. C. SHARP, M.A.)—Find the law of density, varying as the depth, that the centre of pressure of a semi-circle, just immersed with its diameter horizontal, may divide the vertical radius as $m : n$.

Solution.

Let the centre of the circle $x^2 + y^2 = a^2$ be the origin, the horizontal and the vertical radii being the axes of x, y respectively. Let $\rho = \mu y^s$, so that $dp = g\rho dy = \mu g y^s dy$, and, therefore, $p = \frac{\mu g}{s+1} \cdot y^{s+1}$. Manifestly, the

centre of pressure lies on the axis of y; hence

$$\bar{y} = \frac{\iint py\,dy\,dx}{\iint p\,dy\,dx} = \frac{\int_{-a}^{+a}\int_{0}^{(a^2-x^2)^{\frac{1}{2}}} y^{s+2}\,dy\,dx}{\int_{-a}^{+a}\int_{0}^{(a^2-x^2)^{\frac{1}{2}}} y^{s+1}\,dy\,dx}$$

$$= \frac{s+2}{s+3}\cdot\frac{\int_{-a}^{+a}(a^2-x^2)^{\frac{1}{2}(s+3)}dx}{\int_{-a}^{+a}(a^2-x^2)^{\frac{1}{2}(s+2)}dx} = a\cdot\frac{s+2}{s+3}\cdot\frac{\int_0^{\pi}(\sin\theta)^{s+4}d\theta}{\int_0^{\pi}(\sin\theta)^{s+3}d\theta}, \text{ where } x = a\cos\theta,$$

$$= a\cdot\frac{s+2}{s+3}\cdot \mathrm{K}, \text{ suppose.}$$

By hypothesis, we have $\dfrac{\bar{y}}{a-\bar{y}} = \dfrac{m}{n}$,

therefore $$\bar{y} = \frac{ma}{m+n} = a\cdot\frac{s+2}{s+3}\mathrm{K};$$

therefore $$\mathrm{K}\cdot\frac{s+2}{s+3} = \frac{m}{m+n} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

The value of K involving s, depends on the form of s, viz., we have

$$\mathrm{K} = \left\{\frac{2\cdot4\cdot6\ldots(s+3)}{1\cdot3\cdot5\ldots(s+2)}\right\}^2\frac{1}{s+4}\cdot\frac{2}{\pi}$$

when s is odd, and

$$\mathrm{K} = \left\{\frac{1\cdot3\cdot5\ldots(s+3)}{2\cdot4\cdot6\ldots(s+2)}\right\}^2\frac{1}{s+4}\cdot\frac{\pi}{2}$$

when s is even. It is thus theoretically possible to obtain the value of s from (1) in terms of m and n. In the particular case when the density varies directly as the depth, we have

$$s = 1, \quad \mathrm{K} = \left(\frac{2\cdot4}{1\cdot3}\right)^2\cdot\frac{2}{5\pi},$$

and, from (1), we have $$\frac{m}{m+n} = \tfrac{3}{4}\mathrm{K} = \frac{32}{15\pi},$$

which gives $$\frac{m}{n} = \frac{32}{15\pi-32}.$$

If the density varies as the square of the depth, we have $s = 2$, and it is easily found that $\dfrac{m}{n} = \dfrac{15\pi}{64-15\pi}$.

6991. (By J. J. Walker, M.A., F.R.S.)—Let A, B be points on a line determined by $(a, b, c\,\Sigma x, 1)^2 = 0$; C, D two other points on the same line determined by $(a_1, b_1, c_1\,\Sigma x, 1)^2 = 0$; prove that the values of the double, or cross, ratios AC . DB : AB . DC, AD . CB : AB . CD are the roots of $4\mathrm{II}'(2\lambda-1)^2 - \mathrm{J}^2 = 0$, where I, I′, J are the invariants of the quadratic.

Solution.

Let O be the origin, and A, B, C, D the given points; take OA = α, OB = β, OC = γ, OD = δ, so that α, β are the roots of $ax^2+2bx+c=0$, and γ, δ those of $a_1x^2+2b_1x+c_1=0$; therefore we have

$$\alpha+\beta=-\frac{2b}{a},\quad \alpha\beta=\frac{c}{a},\quad \gamma+\delta=-\frac{2b_1}{a_1},\quad \gamma\delta=\frac{c_1}{a_1}.$$

Further, we have $I=b^2-ac$, $I'=b_1^2-a_1c_1$, $J=a_1c+ac_1-2bb_1$.

Calling the given ratios R_1, R_2, and attending to the usual convention of signs, we get

$$R_1=\frac{AC\,.\,DB}{AB\,.\,DC}=\frac{AC\,.\,BD}{AB\,.\,CD}=\frac{(\gamma-\alpha)(\delta-\beta)}{(\beta-\alpha)(\delta-\gamma)},$$

$$R_2=\frac{AD\,.\,CB}{AB\,.\,CD}=-\frac{AD\,.\,BC}{AB\,.\,CD}=-\frac{(\delta-\alpha)(\gamma-\beta)}{(\beta-\alpha)(\delta-\gamma)},$$

where, of course, the symbols α, β, γ, δ have reference only to the lengths, and not to the directions, of the lines. Therefore we get

$$R_1+R_2=\frac{(\gamma-\alpha)(\delta-\beta)-(\delta-\alpha)(\gamma-\beta)}{(\beta-\alpha)(\delta-\gamma)}=\frac{(\beta-\alpha)(\delta-\gamma)}{(\beta-\alpha)(\delta-\gamma)}=1.$$

Again, since $$-R_1R_2=\frac{(\gamma-\alpha)(\delta-\beta)(\delta-\alpha)(\gamma-\beta)}{(\beta-\alpha)^2(\delta-\gamma)^2},$$

we see, by writing $N=(\gamma-\alpha)(\delta-\beta)(\delta-\alpha)(\gamma-\beta)$, $D=(\beta-\alpha)^2(\delta-\gamma)^2$, that

$$N=\{\gamma^2-(\alpha+\beta)\gamma+\alpha\beta\}\{\delta^2-(\alpha+\beta)\delta+\alpha\beta\}$$

$$=\alpha^2\beta^2+\gamma^2\delta^2-2\alpha\beta\gamma\delta-(\alpha+\beta)(\gamma+\delta)(\alpha\beta+\gamma\delta)+\alpha\beta(\gamma+\delta)^2+\gamma\delta(+\beta)^2$$

$$=\frac{1}{a^2a_1^2}\{a^2c_1^2+a_1^2c^2-2aa_1cc_1-4bb_1(ac_1+a_1c)+4acb_1^2+4a_1c_1b^2\}$$

$$=\frac{J^2-4II'}{a^2a_1^2}.$$

Similarly, $$D=(\beta-\alpha)^2(\delta-\gamma)^2=\{(\beta+\alpha)^2-4\beta\alpha\}\{(\delta+\gamma)^2-4\delta\gamma\}$$

$$=\frac{16(b^2-ac)(b_1^2-a_1c_1)}{a^2a_1^2}=\frac{16II'}{a^2a_1^2}.$$

Therefore, $$-R_1R_2=\frac{J^2-4II'}{16II'}.$$

Hence we see that R_1, R_2 are the roots of the quadratic

$$\lambda^2-\lambda(R_1+R_2)+R_1R_2=0,$$

that is, of

$$\lambda^2-\lambda+\frac{4II'-J^2}{16II'}=0,$$

which is easily written in the required form $4II'(2\lambda-1)^2=J^2$.

7212. (By D. Edwardes.)—Prove that

$$(y+z-2u)(z+x-2v)(x+y-2w)-(y+z-2u)(x+y-v-w)^2$$
$$-(z+x-2v)(y+v-w-u)^2-(x+y-2w)(z+w-u-v)^2$$
$$=2(x+u-v-w)(y+v-w-u)(z+w-u-v).$$

Solution.

Let $x+u-v-w=a, \quad y+v-w-u=b; \quad z+w-u-v=c.$

Then we have to show that

$$(b+c)(c+a)(a+b)-(b+c)\,a^2-(c+a)\,b^2-(a+b)\,c^2=2abc,$$

which obviously holds, since the sinister vanishes for $a=0$, $b=0$, $c=0$ separately, and the numerical coefficient is easily determined to be 2.

7538. (By Professor HAUGHTON, F.R.S.)—Show that the law of propagation of heat in a solid sphere is

$$\frac{dv}{dt}=a\left(\frac{d^2v}{dr^2}+\frac{2}{r}\,\frac{dv}{dr}\right).$$

Solution.

The general law of propagation of heat in an isotropic solid

$$\frac{dv}{dt}=a\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right),$$

where a is a function of the conductivity and specific heat of the solid, viz., $a=\mathrm{K}/\mathrm{CD}$. If r be the radius of the spherical surface, we have

$$r^2=x^2+y^2+z^2,$$

and

$$\frac{dv}{dx}=\frac{dv}{dr}\cdot\frac{dr}{dx}=\frac{x}{r}\,\frac{dv}{dr},$$

$$\frac{d^2v}{dx^2}=\frac{x^2}{r^2}\cdot\frac{d^2v}{dr^2}+\frac{dv}{dr}\left(\frac{1}{r}-\frac{x^2}{r^3}\right), \qquad \frac{d^2v}{dy^2}=\frac{y^2}{r^2}\cdot\frac{d^2v}{dr^2}+\frac{dv}{dr}\left(\frac{1}{r}-\frac{y^2}{r^3}\right),$$

$$\frac{d^2v}{dz^2}=\frac{z^2}{r^2}\cdot\frac{d^2v}{dr^2}+\frac{dv}{dr}\left(\frac{1}{r}-\frac{z^2}{r^3}\right).$$

Therefore

$$\nabla^2v=\frac{d^2v}{dr^2}+\frac{2}{r}\,\frac{dv}{dr},$$

whence the equation of motion becomes

$$\frac{dv}{dt}=a\left(\frac{d^2v}{dr^2}+\frac{2}{r}\,\frac{dv}{dr}\right).$$

[For another solution, see Vol. XLI., p. 38.]

7894. (By Professor HUDSON, M.A.)—Prove that, in the steady motion in one plane of a uniform incompressible fluid under the action of natural forces, if u, v be the velocities at x, y, parallel to the axes,

$$v\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)u-u\left(\frac{d^2}{dx^2}+\frac{d^2}{dy^2}\right)v=0.$$

Solution by D. EDWARDES.

An improvement on the solution given by ASÛTOSH MUKHOPÂDYÂY, on p. 142 of his Appendix to Vol. 43, may be effected as follows:—

By GREEN's theorem we have

$$\iint (u\nabla^2 v - v\nabla^2 u)\, d\mathrm{S} = \int \left(u \frac{dv}{dn} - v \frac{du}{dn} \right) ds,$$

where dn is an element of the normal drawn outwards. But

$$\frac{du}{dx} = -\frac{dv}{dy}, \quad \frac{du}{dy} = \frac{dv}{dx},$$

hence
$$u \frac{dv}{dn} - v \frac{du}{dn} = u \left(\frac{du}{dy} \frac{dy}{ds} + \frac{du}{dx} \frac{dx}{ds} \right) + v \left(\frac{dv}{dy} \frac{dy}{ds} + \frac{dv}{dx} \frac{dx}{ds} \right)$$

$$= u \frac{du}{ds} + v \frac{dv}{ds} = \tfrac{1}{2} \frac{dq^2}{ds}$$

(if q is the resultant velocity). Hence $\int \left(u \frac{dv}{dx} - v \frac{du}{dx} \right) ds = 0,$

the integration extending round the boundary, and this requires that

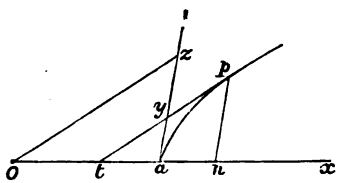
$$u\Delta^2 v - v\Delta^2 u = 0.$$

APPENDIX III.

GEOMETRICAL NOTE ON AN ENVELOPE IN CONNEXION WITH CONFOCAL CONICS.

By R. F. Davis, M.A.

1. Let pn be the ordinate at any point p of a parabola to the diameter ax; and let pt (the tangent at p) meet this diameter in t and the tangent at the vertex of the diameter in y, so that $at = an$ and $ay = \frac{1}{2}pn$. Then if o be any fixed external point in the diameter, and oz be drawn parallel to pt to meet ay produced in z; then



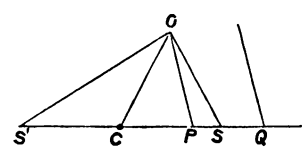
$$az : oa = pn : nt = pn : 2an \text{ and } ay = \tfrac{1}{4}pn;$$

therefore $$ay \,.\, az = oa \,.\, pn / 4an = \text{constant}.$$

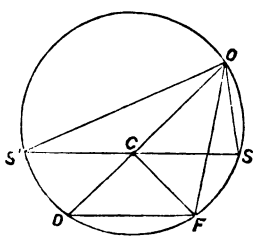
2. Conversely; employing a different notation,—If on a given axis points P, Q be taken such that

$$CP \,.\, CQ = CS^2 = CS'^2$$

(so that P, Q are a pair of points in the fixed involution of which C is the centre and S, S′ the double points or *foci*); then the envelope of the straight line through Q parallel to OP (where O is any fixed external point) is a parabola inscribed in the triangle OSS′. The point of contact of the side SS′ with the curve being at C, OC is the direction of the axis of this parabola.



3. Since OS, OS′ are tangents to the parabola and OC is parallel to the axis, the focus F (which lies on the circumcircle of the triangle OSS′) must also lie on a straight line through O such that $\angle SOF = \angle S'OC$, *i.e.*, the Symmedian line through O.



Hence F is determined as the point on the circumcircle such that OF, SS′ are conjugate chords or OSFS′ is a "harmonic quadrilateral"; O, F are therefore interchangeable. If OC be produced to meet the circumcircle in D; then, since $\angle SOF = \angle S'OD$, $SF = S'D$, $CF = CD = CS^2 / CO$, and CF, CO are equally inclined to CS.

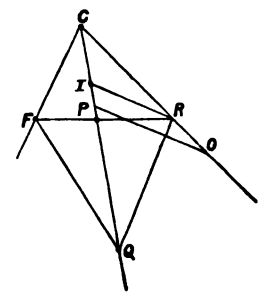
The triangles OCS, SCF are thus similar, and $\angle OSC = \angle SFC$, so that OS touches the circumcircle of the triangle SCF. F may therefore be regarded as the point of intersection of two circles described so as to touch OS, OS′ in S, S′ and to pass through C.

4. It may be easily shown by the "pedal" property of the parabola (or by reciprocation) that, if through any variable point on a fixed tangent to a parabola a straight line be drawn inclined at any constant angle to

the other tangent from that point, its envelope will be a confocal parabola touching the fixed tangent, and having its axis inclined at the constant angle to the axis of the original parabola. This envelope, however, degenerates into a single point—the focus, for a particular value of the constant angle.

5. The property stated in Section 2 may therefore be extended as follows: the envelope of the straight line through Q inclined at a constant angle to OP is, in general, a parabola having the same focus F, touching SS′, and whose axis is inclined at the constant angle to OC.

6. When the constant angle is a right angle, the axis of the parabola is perpendicular to OC. Moreover, when Q coincides with C, P is at infinity on the line SS′, OP is parallel to CS, and a line through C perpendicular to SS′ is a tangent to the envelope. Then, since two perpendicular tangents can be drawn from C to the parabola, C is a point on the directrix and OC is the directrix.

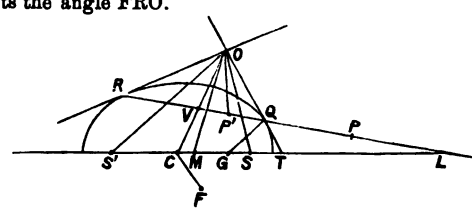


7. On account of its importance, a special investigation may be given for the case when the constant angle is a right angle. Let I be the in-centre and Q an ex-centre of the triangle CFR; so that CIQ bisects the angle at C, and RI, RQ bisect the angles at R.

Then, if from a point O in the line CR, OP be drawn perpendicular to QR, and therefore parallel to RI, the triangle COP is similar to the triangle CRI and consequently to the triangle CQF. Hence CP : CO = CF : CQ, and CP.CQ = CF.CO.

Conversely, if CF, CO be fixed straight lines; CPQ their internal bisector; F, O fixed points, and P, Q variable points such that CP.CQ = CO.CF = constant; then the envelope of a line QR drawn through Q perpendicular to OP is a parabola having F for its focus and CO for its directrix. For QR always bisects the angle FRO.

8. If OQ, OR be tangents to an ellipse, and QR, OM (perpendicular to QR) meet the major axis in L, M respectively, then it is a well-known property that CL.CM = CS².



The problem therefore, to find the envelope of the polar of a given point O with respect to a family of confocal conics (centre C; foci S, S′), reduces to the following:—

If L, M be variable points on the axis such that CL . CM = CS² = CS′², required the envelope of a line through L perpendicular to OM.

This, it has already been shown, in Sections 5—7, is a parabola whose focus F is the point of intersection of the Symmedian line through O of the triangle OSS′ with the circumcircle of the same triangle, and is found by making the ∠FCS = ∠OCS and taking CF . CO = CS². The directrix of this parabola is OC, and the axes of the ellipse are tangents. (Taylor's *Geometry of Conics* 1881, Examples 340, 379, 380.)

9. If the tangent and normal at Q meet the major axis in T and G respectively, we have $CG \cdot CT = CS^2$, and QG is perpendicular to OT. Hence the normals at Q, R touch the same parabola.

10. If these normals intersect in O', F lies on the circumcircle of the triangle O'QR. But this circle passes through O; hence the circumcircle of the triangle OQR passes through F. That is to say, if O be a fixed point, and OQ, OR be tangents to any one of a family of confocal conics, the circumcircle of the triangle OQR belongs to a coaxal system. (*Reprint*, XXII., 4472; XXIII., 4553; &c.)

11. The locus of O' is a fixed straight line through F perpendicular to OF.

12. Moreover, since OC bisects both SS' in the triangle OSS' and QR in the triangle OQR, and $\angle QOS = \angle ROS'$, these triangles (besides having a common circumcircle) have also a common Symmedian line OF through O. Hence, if V be the middle point of QR, $OQ \cdot OR = OV \cdot OF$. (TAYLOR, 948.) For, in Section 3, we see the triangles OFS, OS'C are obviously similar.

13. If upon QR points P, P' be taken, such that $VP \cdot VP' = VQ^2 = VR^2$, then QR is harmonically divided by P, P' and OP' is the polar of P. The envelope of a straight line through P parallel to its polar OP' is, by what has preceded, a parabola inscribed in the triangle OQR, touching QR at its middle point V, and whose focus is the point where the Symmedian line of the triangle OQR meets the circumcircle of the same triangle. This is the point F previously found, and is the same for every confocal.

14. Similarly, if through P any line be drawn making a constant angle with OP' (the polar of P), its envelope is a parabola touching QR and having F for its focus. This parabola degenerates into the point F (see Section 4) when the constant angle $= \angle OQF = \angle CVQ$ = angle between QR and the diameter CO conjugate to it. (Professor WOLSTENHOLME, *Educational Times*, March 1885, No. 8045.) For in Section 3, since $\angle CSF = \angle COS$, to each add $\angle CSO$; then $\angle OSF = \angle SCD$.

15. When, however, the constant angle is a right angle, the normals at Q, R (as well as QR) are tangents to the parabola, which in this case coincides with the envelope of QR discussed in Section 8, and is the same for every confocal.

16. As in Section 9, if OU be normal to the ellipse at U the tangent to the ellipse at U also touches this parabola. Hence (theoretically at least) we are furnished with a geometrical construction for determining the four points on any ellipse the normals at which pass through a given point O. For describing the parabola-envelope belonging to O, the required points are the points of contact with the ellipse of the four common tangents to the two curves (TAYLOR, 990).

17. If, from any point in a given fixed straight line through the focus of a parabola, tangents be drawn to the curve, then it may be shown that the envelope of the line joining the feet of the perpendiculars on these tangents from a given fixed point on the directrix is another parabola. The focus of the envelope is the foot of the perpendicular from the fixed point on the fixed line; and the directrix passes through the fixed point and makes the same angle with this perpendicular as the directrix of the original parabola makes with the focal distance of the fixed point.

Hence the line joining the feet of the perpendiculars from the centre on the two normals corresponding to any one confocal envelopes a fixed parabola. If CK be perpendicular upon FH (the fixed straight line perpendicular to OF, upon which the normals always intersect), then K is the focus of the envelope; and if upon OF a point E be taken such that $OE \cdot OF = OC^2$, then the directrix is CE. This result may be reconciled

with Professor WOLSTENHOLME's form of the equation to the envelope (*Educational Times*, May 1886, No. 8559).

NOTE* BY "ASPARAGUS."

(*a*) "Given a conic whose foci are S, S′ and a point O; from any point P on the polar of O is drawn a straight line inclined at a given angle to the polar of P; prove that the envelope of this straight line is a parabola, whose focus O′ is such that the conic, passing through O, O′, and the points of contact of the tangents from OO′ to the given conic, is a circle."

(*b*) "OP, OQ being two tangents to a conic, V the middle point of PQ, the circles touching the conic in P, Q and passing through V intersect again in the point O′ of the question."

First take the case where the straight line is drawn perpendicular to the polar of P; then this straight line and the polar of P are two conjugate straight lines with respect to the conic, and, if Q be their point of intersection, QO, QP are two conjugates at right angles, and are, therefore, equally inclined to the focal distances SQ, S′Q. [This shows that the system of straight lines depends only on S, S′, and O, and therefore that the envelope will be the same for all confocals.] Through each point of the polar of O, one and only one of these straight lines can be drawn, but in one position the straight line coincides with the polar of O; hence through each point P two tangents can be drawn to the envelope, which is therefore a conic. When P is on the axis, the straight line coincides with that axis, hence the conic envelope touches the axes; when P is at the point of contact of either tangent from O, the straight line is the normal; hence the conic envelope touches the axes, the polar of O, and the normals at the points where the polar of O meets the conic; and these, it is well known, touch a parabola whose directrix is CO and whose focus O′ is such that CO, CO′ are equally inclined to the major axis on opposite sides, and $CO \cdot CO' = CS^2$. (When P is at infinity, the perpendicular to its polar becomes the line at infinity, which proves that the envelope is a parabola). But the conic through (XY), (X′Y′), and the points of contact of the tangents from them to the conic $a^2y^2 + b^2x^2 = a^2b^2$, is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{XX'}{a^2} + \frac{YY'}{b^2} - 1\right) = \left(\frac{xX}{a^2} + \frac{yY}{b^2} - 1\right)\left(\frac{xX'}{a^2} + \frac{yY'}{b^2} - 1\right);$$

and this is a circle if $XY' + X'Y = 0$, and $XX' - a^2 = YY' - b^2$; or if $\frac{X}{X'} = -\frac{Y}{Y'} = \frac{X^2+Y^2}{a^2-b^2}$, which gives the same relation between O, O′ as stated above. Hence the envelope of the straight line drawn through P perpendicular to its polar with respect to a given conic is a parabola, which can be constructed from S, S′, O alone; and therefore the envelope will be the same for all confocals, although there would then appear *a priori* to be two parameters involved. The envelope, in the case when any given angle is made, now reduces to the well known one: From any point (P) on a fixed tangent to a given parabola, is drawn a straight line making a constant angle with the 2nd tangent through P, the envelope is another parabola having the same focus (by quite elementary geometry).

* [Mr. DAVIS's attention was first called to this subject by reading a proof of the following note by ASPARAGUS, in reference to the questions (*a*, *b*), whereof (*b*) is a simple deduction from (*a*), and (*a*) has been solved somewhere in our volumes (though we have not been able at once to find and add the reference), though not so satisfactorily as in ASPARAGUS's appended solution].

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